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ELEMENTARY INTEGRALS

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1. Introduction. This paper is concerned with the following problem: What are the conditions that an elementary function possess an elementary indefinite integral? This question was first successfully treated by J. Liouville in a series of papers which appeared between 1833 and 1840. The work of Liouville and subsequent investigators is presented in Ritt's monograph [1]. Although the problem is one which arises very naturally in the first course in calculus, its solution, as presented in Ritt [1], requires mathematical techniques of a considerably more advanced level. It is the purpose of this paper to show that a more elementary treatment is possible if one uses a more restrictive definition of elementary function.

2. Elementary functions. In the calculus an elementary function is one which can be constructed out of the variable x by repeated use of: (a) algebraic operations, (b) the taking of exponentials, (c) the taking of logarithms. To make "algebraic operation" more explicit we recall the notion of algebraic function. One says that y is an algebraic function of x if it is defined by an irreducible relation

$$a_0y^n + a_1y^{n-1} + \cdots + a_n = 0$$

where $a_0, a_1, \cdots, a_n$ are polynomials in x, and a_0 is not identically zero.

It is immediately clear from this definition why one must use more advanced techniques of analysis in treating the problem under consideration. For the algebraic functions are, in general, multiple-valued, and one is naturally led to consideration of branch points, Riemann surfaces, and related topics in theory of functions of a complex variable.

It is also clear that one may avoid this complication if one is willing to restrict the nature of "elementary function" as regards "algebraic operation", if one requires that y be elementary in x only if one has an *explicit* relationship $y=f(x)$ in which the operator f is constructed from the three basic types, *viz.* algebraic, exponential, logarithmic. In this case one means by an algebraic operation one which involves only the extracting of root in addition to the usual rational operations. However, one may even dispense with extraction of root, since such an operation can be replaced by a combination of rational, exponential, and logarithmic operations, *i.e.*,

$$x^{1/n} = e^{(1/n)\log x}.$$

Henceforth, we shall consider "elementary function" as being understood in this narrower sense.

3. The construction of an elementary function. In order to state more explicitly the way in which an elementary function is constructable we shall employ Ostrowski's method of field extensions (see Ritt, p. 42). This device offers us a two-fold advantage: (i) it is simpler than the structure given by Ritt ([1], p. 10–11); (ii) one can carry out the theory for a "generalized" concept of elementary function with no additional difficulty.

At the core of this method is the concept of a differential field, *i.e.*, a class of functions which form an algebraic field and is closed with respect to the operation of differentiation. To be more specific, we shall say that a class $\mathfrak{F}$ of real valued functions of a real variable, each of which is piecewise differentiable on a common finite interval I, is a *differential field* provided that

(i) $\mathfrak{F}$ contains all the constant functions

(ii) $\alpha, \beta \in \mathfrak{F} \Rightarrow \alpha+\beta, \alpha\cdot\beta \in \mathfrak{F}$

(iii) $\alpha \in \mathfrak{F} \Rightarrow \alpha' \in \mathfrak{F}$

(iv) $\alpha \in \mathfrak{F} \Rightarrow 1/\alpha \in \mathfrak{F}$ unless $\alpha \equiv 0$ on I.

We note that (iii), (iv) imply that the zeros of any non-constant member of $\mathfrak{F}$ are isolated.

We now introduce the concept of *field extension*. If $\mathfrak{F}$ is a differential field and θ is a piecewise differentiable function not in $\mathfrak{F}$, then the class of rational functions of θ with coefficients in $\mathfrak{F}$ is a field. If it is also a differential field, then we call this field a *simple extension* of $\mathfrak{F}$ and denote it $\mathfrak{F}(\theta)$.

We generalize the last. $\mathfrak{F}(\theta_1, \cdots, \theta_n)$ is a simple extension of $\mathfrak{F}(\theta_1, \cdots, \theta_{n-1})$. We call $\mathfrak{F}(\theta_1, \cdots, \theta_n)$ an extension of $\mathfrak{F}$.

It is clear now how one may describe a structure for a given elementary function. One starts from the field of rational functions of x and makes the necessary extensions in each case by adjoining the appropriate exponential or logarithm. For example, suppose that

$$\alpha(x) = e^x + \log (x + e^x).$$

If $\mathfrak{F}$ is the field of rational functions of x then

$$\alpha \in \mathfrak{F}(e^x, \log (x + e^x)).$$

4. Theorems on differential fields. In the following we shall assume that $\mathfrak{F}$ is a given differential field. Identities will be understood to be identities on the interval I on which the functions of the field are defined.

THEOREM 1.

$$\left.\begin{array}{ll} \text{(i)} & \theta \notin \mathfrak{F} \\ \text{(ii)} & \theta' \in \mathfrak{F} \\ \text{(iii)} & \beta_0, \cdots, \beta_n \in \mathfrak{F} \\ \text{(iv)} & \beta_0\theta^n + \beta_1\theta^{n-1} + \cdots + \beta_n \equiv 0 \end{array}\right] \Rightarrow \beta_0 \equiv \beta_1 \equiv \cdots \beta_n \equiv 0.$$

Proof. If false, then we can assume $\beta_0 \neq 0$ and that n is minimal. We can then rewrite the equation of (iv) in the form

$$\theta^n + \gamma_1\theta^{n-1} + \cdots + \gamma_n = 0$$

where

$$\gamma_i = \frac{\beta_i}{\beta_0} \in \mathfrak{F}.$$

Differentiating, we get

$$(n\theta' + \gamma_1')\theta^{n-1} + [\gamma_1(n-1)\theta' + \gamma_2']\theta^{n-2} + \cdots + \gamma_n' = 0.$$

All coefficients must vanish; otherwise n is not minimal. Thus $n\theta' = -\gamma_1'$. Therefore, $n\theta = c - \gamma_1$. But this contradicts (i).

THEOREM 2.

(i) $\alpha \in \mathfrak{F}$
(ii) $e^{p\alpha} \notin \mathfrak{F}$, $p = 1, 2, \cdots, n$
(iii) $\beta_0, \cdots, \beta_m \in \mathfrak{F}$, *where* $m < n$
(iv) $\beta_0\theta^m + \beta_1\theta^{m-1} + \cdots + \beta_m \equiv 0$, *where* $\theta = e^{\alpha}$

$\Rightarrow \beta_0 \equiv \beta_1 \equiv \cdots \equiv \beta_m \equiv 0.$

Proof. Suppose $\beta_0 \neq 0$. Then, letting $\gamma_i = \beta_i/\beta_0 \in \mathfrak{F}$, we have

$$\theta^m + \gamma_1\theta^{m-1} + \cdots + \gamma_m = 0. \tag{A}$$

Differentiating, we get

$$m\alpha'\theta^m + [\gamma_1' + (m-1)\alpha'\gamma_1]\theta^{m-1} + \cdots + \gamma_m' = 0. \tag{B}$$

For any fixed value of x, the m values of θ determined by (A) also satisfy (B). Thus the left-hand sides of these equations must be equivalent as polynomials in θ. Hence $m\alpha'\gamma_1 = \gamma_1' + (m-1)\alpha'\gamma_1$, $m\alpha'\gamma_2 = \gamma_2' + (m-2)\alpha'\gamma_2$, $\cdots$, $m\alpha'\gamma_m = \gamma_m'$. If $\beta_k \neq 0$, then $k\alpha' = \gamma_k'/\gamma_k$. And so $k\alpha = c + \log|\gamma_k|$. Thus $e^{k\alpha} = a\gamma_k \in \mathfrak{F}$. This contradicts (ii).

THEOREM 3.

(i) $\alpha, e^{n\alpha} \in \mathfrak{F}$
(ii) $e^{p\alpha} \notin \mathfrak{F}$, $p = 1, 2, \cdots, n-1$
(iii) $\beta_0, \cdots, \beta_{n-1} \in \mathfrak{F}$
(iv) $\theta = e^{\alpha}$

$\Rightarrow$ *There exist* $\gamma_0, \cdots, \gamma_{n-1}$, *not all zero,* $\in \mathfrak{F}$ *such that*

$$\left[\sum_{k=0}^{n-1} \beta_k\theta^k\right]\left[\sum_{k=0}^{n-1} \gamma_k\theta^k\right] \in \mathfrak{F}.$$

Proof.

$$\left[\sum_{k=0}^{n-1}\beta_k\theta^k\right]\left[\sum_{k=0}^{n-1}\gamma_k\theta^k\right] = \sum_{k=0}^{2n-2}\theta^k\sum_{j=0}^{k}\beta_{k-j}\gamma_j$$

$$= \sum_{k=0}^{n-1}\theta^k\sum_{j=0}^{k}\beta_{k-j}\gamma_j + \sum_{k=n}^{2n-2}\theta^k\sum_{j=0}^{k}\beta_{k-j}\gamma_j$$

$$= \sum_{k=0}^{n-1}\theta^k\left[\sum_{j=0}^{k}\beta_{k-j}\gamma_j + \theta^n\sum_{j=k+1}^{n-1}\beta_{k+n-j}\gamma_j\right].$$

If this product is to be in $\mathfrak{F}$, then, by Theorem 2,

$$\sum_{j=0}^{k}\beta_{k-j}\gamma_j + \theta^n\sum_{j=k+1}^{n-1}\beta_{k+n-j}\gamma_j \equiv 0$$

for $k=1, 2, \cdots, n-1$.

Now suppose that β_m is the last member of $\beta_0, \cdots, \beta_{n-1}$ which is not identical to zero. Each γ_j except γ_{n-m} occurs in at least one equation with the coefficient β_m. For if $k-j=m$, then $j=k-m$ has values $0, 1, 2, \cdots, n-1+m$, while if $k+n-j=m$, then $j=k+n-m$ has values $n-m+1, n-m+2, \cdots, n-1$. The coefficients of γ_{n-m} are obtained from $j=n-m$. Thus $k-j=k+m-n$ and $k+n-j=k-m$. And so the following β's multiply γ_{n-m}: $\beta_0, \beta_1, \cdots, \beta_{n-1-m}, \beta_{m-1}, \beta_{m-2}, \cdots, \beta_0$. We can assume one of these is not zero, for if all were, then the result of the theorem would obviously hold.

We thus have a system of $n-1$ equations in n unknowns $\gamma_0, \cdots, \gamma_{n-1}$. If the rank of the matrix of this system is $n-r$, we may transpose r of the γ's leaving a non-vanishing coefficient determinant on the left. We can easily choose the γ's on the right so that not all right-hand terms are zero. We then solve for the remaining γ's.

THEOREM 4.

$$\left.\begin{array}{ll} \text{(i)} & \alpha,\ e^{n\alpha} \in \mathfrak{F} \\ \text{(ii)} & e^{p\alpha} \notin \mathfrak{F},\ p = 1, 2, \cdots, n-1 \\ \text{(iii)} & \beta_0, \beta_1, \cdots, \beta_m \in \mathfrak{F} \\ \text{(iv)} & P(\theta) = \sum_{k=0}^{m}\beta_k\theta^k \equiv 0,\ \textit{where}\ \theta = e^{\alpha} \\ \text{(v)} & \omega^n = 1 \end{array}\right] \Rightarrow P(\omega\theta) \equiv 0.$$

Proof. Let $[m/n]=r$. Then

$$P(\theta) = \sum_{k=0}^{m}\beta_k\theta^k = \sum_{k=0}^{n-1}\beta_k\theta^k + \sum_{k=n}^{2n-1}\beta_k\theta^k + \cdots + \sum_{k=rn}^{m}\beta_k\theta^k$$

$$= \sum_{k=0}^{n-1}[\beta_k + \theta^n\beta_{k+n} + \cdots + \theta^{rn}\beta_{k+rn}]\theta^k \equiv 0.$$

By Theorem 2, $\beta_k+\theta^n\beta_{k+n}+\cdots+\theta^{rn}\beta_{k+rn}\equiv 0$, $k=1, 2, \cdots, n-1$. Since these last equations are unchanged by the substitution of $\omega\theta$ for θ, it follows that $P(\omega\theta)\equiv 0$.

THEOREM 5.

$$\left.\begin{array}{ll} \text{(i)} & \alpha\in\mathfrak{F} \\ \text{(ii)} & \theta=e^{\alpha}\notin\mathfrak{F} \\ \text{(iii)} & \beta\in\mathfrak{F}(\theta) \\ \text{(iv)} & \beta\notin\mathfrak{F} \end{array}\right\}\Rightarrow\beta'\notin\mathfrak{F}.$$

Proof. Case 1. If $e^{p\alpha}\notin\mathfrak{F}$, $p=1, 2, \cdots$, then it follows from Theorem 2 that β can be written uniquely in the form

$$\beta=P(\theta, x)+\frac{Q(\theta, x)}{R(\theta, x)}$$

where P, Q, R are polynomials in θ with coefficients in $\mathfrak{F}$, the degree of Q being less than that of R and $(Q, R)=1$, *i.e.*, no common factor in $\mathfrak{F}(\theta)$.

Differentiating, we have

$$\beta'=P_x+P_\theta\theta\alpha'+\frac{RQ_x-QR_x+\theta\alpha'(RQ_\theta-QR_\theta)}{R^2}.$$

If $\beta'\in\mathfrak{F}$, it follows from Theorem 2 that $P_x+P_\theta\theta\alpha'\in\mathfrak{F}$. Thus, if

$$P=\sum_{k=0}^{n}\rho_k\theta^k,$$

then

$$\sum_{k=0}^{n}[\rho_k'+k\rho_k\alpha']\theta^k\in\mathfrak{F}.$$

Using Theorem 2 we have $\rho_n'+n\rho_n\alpha'=0$. And so $n\alpha'=-\rho_n'/\rho_n$, giving $n\alpha=c-\log|\rho_n|$. Thus $e^{n\alpha}\in\mathfrak{F}$; therefore $n=0$.

We now have

$$R^2(\beta'-P_x)=RQ_x-QR_x+\theta\alpha'(RQ_\theta-QR_\theta).$$

As polynomials in θ, the degree on the left is greater than that on the right. In view of Theorem 2, this is impossible.

Case 2. If $e^{p\alpha}\notin\mathfrak{F}$, $p=1, 2, \cdots, n-1$, but $e^{n\alpha}\in\mathfrak{F}$, then, by Theorem 3, every quotient of polynomials in θ of degree less than n can be written as a product of such polynomials and hence eventually as a polynomial in θ of degree less than n. Thus we may write

$$\beta=\beta_k\theta^k+\beta_{k-1}\theta^{k-1}+\cdots+\beta_0 \text{ where } k<n.$$

Differentiating, we get

$$\beta' = (\beta_k' + k\alpha'\beta_k)\theta^k + \cdots + \beta_0'.$$

If $\beta' \in \mathfrak{F}$, then, by Theorem 2, $\beta_k' + k\alpha'\beta_k = 0$. This leads to $e^{k\alpha} = a/\beta_k \in \mathfrak{F}$, which contradicts our assumption.

THEOREM 6.

(i) $\alpha \in \mathfrak{F}$
(ii) $\theta = \log |\alpha| \notin \mathfrak{F}$
(iii) $\beta \notin \mathfrak{F}$ but $\beta \in \mathfrak{F}(\theta)$
(iv) $\beta' \in \mathfrak{F}$

$\Rightarrow$ *There exists* $\gamma \in \mathfrak{F}$ *and a constant* c *such that* $\beta = \gamma + c \log |\alpha|$.

Proof. From Theorem 1 we see that β has a unique representation in the form $\beta = P + Q/R$ where P, Q, R are polynomials in θ with coefficients in $\mathfrak{F}$, $(Q, R) = 1$, and the degree of Q is less than that of R. Differentiating, we get

$$\beta' = P_x + \frac{\alpha'}{\alpha} P_\theta + \frac{RQ_x - QR_x + \dfrac{\alpha'}{\alpha}(RQ_\theta - QR_\theta)}{R^2} \in \mathfrak{F}.$$

It follows from Theorem 1 that $P_x + (\alpha'/\alpha)P_\theta \in \mathfrak{F}$. Thus, if $P = \rho_n\theta^n + \cdots + \rho_0$, then

$$\rho_n'\theta^n + \left(\rho_{n-1}' + n\rho_n \frac{\alpha'}{\alpha}\right)\theta^{n-1} + \cdots + \rho_0' \in \mathfrak{F}.$$

And so

$$\rho_n' = \rho_{n-1}' + n\rho_n \frac{\alpha'}{\alpha} = \rho_{n-2}' + (n-1)\rho_{n-1}\frac{\alpha'}{\alpha} = \cdots = \rho_1' + 2\rho_2\frac{\alpha'}{\alpha} = 0.$$

Since $\rho_n' = 0$, ρ_n is a constant. But then $\rho_{n-1} = c - n\rho_n \log \alpha$. Therefore, $\rho_n = 0$. And so ρ_{n-1} is constant. Continuing in this way, we see that $\rho_n = \rho_{n-1} = \cdots = \rho_2 = 0$ and ρ_1 is a constant. Thus

$$\beta' = \rho_1 \frac{\alpha'}{\alpha} + \rho_0' + \frac{RQ_x - QR_x + \dfrac{\alpha'}{\alpha}(RQ_\theta - QR_\theta)}{R^2}.$$

Rearranging, we get

$$\left(\beta' - \rho_1\frac{\alpha'}{\alpha} - \rho_0'\right)R^2 = RQ_x - QR_x + \frac{\alpha'}{\alpha}(RQ_\theta - QR_\theta)$$

which, by Theorem 1, is possible only if the degree of R is zero. Thus $\beta = \rho_0 + \rho_1 \log |\alpha|$, where $\rho_0 \in \mathfrak{F}$ and ρ_1 is constant.

THEOREM 7.

(i) $\alpha \in \mathfrak{F}$

(ii) $\int \alpha \in \mathfrak{F}(\theta_1, \cdots, \theta_n)$

(iii) *each* θ_i *is either an exponential or a logarithm*

$\Rightarrow$ *There exist* $\beta_0, \cdots, \beta_n \in \mathfrak{F}$ *and constants* $c_1, \cdots, c_n$ *such that*

$$\int \alpha = \beta_0 + c_1 \log \beta_1 + \cdots + c_n \log \beta_n.$$

Proof. Suppose $n=1$. Then the result follows from Theorems 5, 6. Assume the theorem is true for $n=1, 2, \cdots, s$. We consider the case $n=s+1$. Let $\mathfrak{G}=\mathfrak{F}(\theta_1)$. Then $\alpha \in \mathfrak{G}$ and $\int \alpha \in \mathfrak{G}(\theta_2, \cdots, \theta_{s+1})$. Thus

$$\int \alpha = \beta_0 + c_1 \log \beta_1 + \cdots + c_s \log \beta_s$$

where $\beta_0, \cdots, \beta_s \in \mathfrak{G}$ and $c_1, \cdots, c_s$ are constants. Let $\theta=\theta_1$. Then we have

$$\int \alpha = \beta_0(\theta, x) + \sum c_i \log \beta_i(\theta, x)$$

and

$$\alpha = \beta_{0,\theta}\theta' + \beta_{0,x} + \sum \frac{c_i}{\beta_i} [\beta_{i,\theta}\theta' + \beta_{i,x}].$$

Case 1. Suppose $\theta = e^{\gamma}$, where $\gamma \in \mathfrak{F}$ and $e^{p\gamma} \notin \mathfrak{F}$, $p=1, 2, \cdots$. Then $\theta' = \theta\gamma'$ and we have

$$\alpha = \beta_{0,\theta}\theta\gamma' + \beta_{0,x} + \sum \frac{c_i}{\beta_i} [\beta_{i,\theta}\theta\gamma' + \beta_{i,x}].$$

This must be an identity in θ by Theorem 2, since each β_i is a rational function of θ. If we replace θ by $\mu\theta$ we get

$$\alpha = \beta_{0,\theta}(\mu\theta, x)\mu\theta\gamma' + \beta_{0,x}(\theta\mu, x) + \sum \frac{c_i}{\beta_i(\mu\theta, x)} [\beta_{i,\theta}(\mu\theta, x)\mu\theta\gamma' + \beta_{i,x}(\mu\theta, x)]$$

which is the derivative of

$$\beta_0(\mu\theta, x) + \sum c_i \log \beta_i(\mu\theta, x).$$

Thus

$$\beta_0(\mu\theta, x) + \sum c_i \log \beta_i(\mu\theta, x) = \beta_0(\theta, x) + \sum c_i \log \beta_i(\theta, x) + f(\mu).$$

Differentiating with respect to μ and setting $\mu=1$ we have

$$\beta_{0,\theta}(\theta, x)\theta + \sum \frac{c_i}{\beta_i(\theta, x)} \beta_{i,\theta}(\theta, x)\theta = f'(1).$$

Therefore

$$\beta_{0,\theta}(\theta, x) + \sum \frac{c_i}{\beta_i(\theta, x)} \beta_{i,\theta}(\theta, x) = \frac{f'(1)}{\theta}.$$

This is an identity in θ. Thus

$$\beta_0(\theta, x) + \sum c_i \log \beta_i(\theta, x) = f'(1)[\gamma - \log \theta_0] + \beta_0(\theta_0, x) + \sum c_i \log \beta_i(\theta_0, x).$$

And so we can express $\int\alpha$ in the desired form.

Case 2. Suppose $\theta = \log |\gamma|$. Then

$$\alpha = \beta_{0,\theta} \frac{\gamma'}{\gamma} + \beta_{0,\theta} + \sum \frac{c_i}{\beta_i}\left[\beta_{i,\theta} \frac{\gamma}{\gamma} + \beta_{i,x}\right].$$

By Theorem 1 this must be an identity in θ. We replace θ by $\theta+\mu$. Thus

$$\alpha = \beta_{0,\theta}(\theta+\mu, x) \frac{\gamma'}{\gamma} + \beta_{0,x}(\theta+\mu, x) + \sum \frac{c_i}{\beta_i}\left[\beta_{i,\theta}(\theta+\mu, x) \frac{\gamma'}{\gamma} + \beta_{i,x}(\theta+\mu, x)\right],$$

which is the derivative of

$$\beta_0(\theta+\mu, x) + \sum c_i \log \beta_i(\theta+\mu, x).$$

And so

$$\beta_0(\theta+\mu, x) + \sum c_i \log \beta_i(\theta+\mu, x) = \beta_0(\theta, u) + \sum c_i \log \beta_i(\theta, x) + f(\mu).$$

Differentiating with respect to μ and setting $\mu=0$ we get

$$\beta_{0,\theta}(\theta, x) + \sum \frac{c_i}{\beta_i(\theta, x)} \beta_{i,\theta}(\theta, x) = f'(0).$$

This is an identity in θ. Thus

$$\int \alpha = \beta_0(\theta, x) + \sum c_i \log \beta_i(\theta, x) = f'(0)(\theta - \theta_0) + \beta_0(\theta_0, x) + \sum c_i \log \beta_i(\theta_0, x).$$

And so $\int\alpha$ is of the desired form.

Case 3. Suppose $\theta = e^\gamma$ where $\gamma \in \mathfrak{F}$ and $e^{p\gamma} \notin \mathfrak{F}$, $p=1, 2, \cdots, n-1$, but $e^{n\gamma} \in \mathfrak{F}$. We get

$$\alpha = \beta_{0,\theta}\theta\gamma' + \beta_{0,x} + \sum \frac{c_i}{\beta_i}\left[\beta_{i,\theta}\theta\gamma' + \beta_{i,x}\right]$$

where, in view of Theorem 3, each β_i is a polynomial in θ of degree less than n.

By Theorem 4 we can replace θ by $\omega^k\theta$, $k=0, 1, \cdots, n-1$, where $\omega^n=1$, in

the last. We thus have

$$\int \alpha = \beta_0(\omega^k\theta, x) + \sum c_i \log \beta_i(\omega^k\theta, x), \qquad k = 0, 1, \cdots, n-1.$$

And so

$$\int \alpha = \frac{1}{n}\sum_{k=0}^{n-1} \beta_0(\omega^k\theta, x) + \frac{1}{n}\sum c_i \log\left[\prod_{k=0}^{n} \beta_i(\omega^k\theta, x)\right].$$

But the expressions $\sum\beta_0(\omega^k\theta, x)$, $\prod\beta_i(\omega^k\theta, x)$ are symmetric polynomials in the quantities $\theta, \omega\theta, \cdots, \omega^{n-1}\theta$. Since these are the roots of $\theta^n - \delta = 0$, where $\delta \in \mathfrak{F}$, it follows that the expressions are rational in δ, and hence are in $\mathfrak{F}$. This gives $\int\alpha$ the desired form.

5. Applications. Theorem 7 of the last section is our version of the Liouville-Ostrowski Theorem ([1], p. 42). Much of the subsequent development in Ritt's monograph may now be carried through with suitable modifications. For example, we have the following version of a result due to Liouville ([1], p. 47):

THEOREM 8.

(i) $\alpha, \beta \in \mathfrak{F}$

(ii) $e^{n\alpha} \notin \mathfrak{F}$, $n = 1, 2, \cdots$,

(iii) $\gamma = \int e^{\alpha}\beta$ *is elementary with respect to* $\mathfrak{F}$, *i.e.,* γ *is contained in an extension of* $\mathfrak{F}$

$\Rightarrow$ *There exists* $\delta \in \mathfrak{F}$ *such that* $\gamma = e^{\alpha}\delta + constant.$

Proof. Let $e^{\alpha} = \theta$. Then $e^{\alpha}\beta = \theta\beta \in \mathfrak{F}(\theta)$. Thus, by Theorem 7,

$$\gamma = \lambda_0(\theta, x) + \sum_{i=0}^{k} c_i \log \lambda_i(\theta, x)$$

where $\lambda_i \in \mathfrak{F}(\theta)$, $i = 0, 1, 2, \cdots, k$. Differentiating, we have

$$\theta\beta = \lambda_{0,\theta}\theta\alpha' + \lambda_{0,x} + \sum \frac{c_i}{\lambda_i}\left[\lambda_{i,\theta}\theta\alpha' + \lambda_{i,x}\right].$$

By Theorem 2 this must be an identity in θ. We replace θ by $\mu\theta$ and integrate. Then

$$\mu\gamma = \lambda_0(\mu\theta, x) + \sum c_i \log \lambda_i(\mu\theta, x) + f(\mu).$$

Differentiating with respect to μ and setting $\mu = 1$ gives

$$\gamma = \theta\lambda_{0,\theta}(\theta, x) + \sum \frac{c_i\theta\lambda_{i,\theta}(\theta, x)}{\lambda_i} + f'(1).$$

Let $\omega(\theta, x) = \lambda_0(\theta, x) + \sum c_i \log \lambda_i(\theta, x)$. We have

$$\omega(\theta, x) = \theta\omega_{,\theta}(\theta, x) + f'(1).$$

Hence

$$\gamma = \omega(\theta, x) = \frac{\theta}{\theta_0}\left[\omega(\theta_0, x) - f'(1)\right] + f'(1).$$

And so γ has the desired form.

Consider, for example, the case of $\gamma = \int e^{x^2} dx$. Theorem 8 applies where $\mathfrak{F}$ is the field of rational functions of x. If γ is to be elementary, then $\gamma = e^{x^2}\delta + \text{constant}$. But $\gamma' = e^{x^2} = e^{x^2}\delta' + 2xe^{x^2}\delta$. Hence $\delta' + 2x\delta = 1$. Now δ is rational in x and thus can be written in the form $\delta = P + Q/R$, where P, Q, R are polynomials in x, $(Q, R) = 1$, Q of degree less than R. We have

$$P' + \frac{RQ' - QR'}{R^2} + 2xP + \frac{2xQ}{R} - 1 \equiv 0.$$

This is not possible for any polynomials P, Q, R (since coefficients must vanish). And so $\int e^{x^2} dx$ is not elementary.

6. Trigonometric integrals. The restriction to real functions leaves out the integrands involving trigonometric functions, these being included in the complex case in their exponential form. But it is clear that, in order to treat such an integrand, one may replace $\sin x$ by $(1/a)\sinh ax$ and examine the resulting integrand. If its elementary integral is obtained, we replace a by $\sqrt{-1}$ to obtain an integral for the original expression.

Reference

[1] J. F. Ritt, Integration in Finite Terms, Columbia University Press, 1948.

INTEGRAL APPROXIMATE SOLUTIONS OF SYSTEMS OF LINEAR EQUATIONS

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1. Statement of the problem. Consider

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \qquad (i = 1, \cdots, m). \tag{1}$$

This is a system of m linear equations in n unknowns, where m and n are arbitrary positive integers. The coefficients, a_{ij}, are real. The case of complex coefficients can be disposed of by doubling the number of equations.

As several kinds of solutions will be mentioned, the following terms are used:

e-solution is an exact solution;

a-solution is an approximate solution;

i-solution is an integral solution;

l-solution is a least square solution.

There would also be the combinations: *ia*-solution, meaning an approximate solution where the unknowns have integral values; *lia*-solution, meaning an *ia*-solution which compared with other *ia*-solutions has the sum of the squares of its deviations a minimum.

Many different situations may arise. For example, in 3-space, the equations of the system may represent planes which form a closed polyhedron, a prism, they may all pass through a point or a straight line, or they may all be parallel.

Following are two interesting systems, each of one equation in two unknowns:

System A. $y = \sqrt{2}x - 0.5.$

System B. $y = 1.1x - \sqrt{2}.$

System A has, among others, the two *ia*-solutions $(-1, -1)$, and $(-35, -49)$. The former is more convenient to use, since the values of x and y are small integers. The latter has much the smaller error. Further *ia*-solutions with still larger values of x and y, but with still smaller errors, exist. Similar situations in higher dimensions are easily imagined by the reader. It will be shown that each of the two *ia*-solutions cited can be obtained as an *lia*-solution of a so-called Complete System.

System B has the *ia*-solutions $x=4+10q$, $y=3+11q$, for any integral value of q. These all have the same error, and all other *ia*-solutions have larger error, no matter how large values of x and y may be considered.

The method to be proposed for finding *ia*-solutions will be the same for any system of linear equations. It is not necessary to determine in advance which kind of a system you have. After you have carried out the solution you will know whether the system is like System A or System B, or unlike both of them.

The practical computer will object that he is not interested in *ia*-solutions. He may want the values of x_i carried out to 4 decimal places. Then he should multiply the constant terms of the system by 10^4, and proceed to find *ia*-solutions, which later can be divided by 10^4. In this way, he can be sure that his results are correct to 4 decimal places.

The procedure requires the adjoining of certain auxiliary equations, forming the so-called Complete System. The *lia*-solution of the latter depends on finding the least value of a positive quadratic form for integral values of the variables. The method of handling the quadratic form, devised by Minkowski, will be briefly outlined.

2. System of congruences. The method will also solve systems of congruences. Such a system can be converted into a system of linear equations, with one new unknown for each congruence. For example, the system of congruences

$$\begin{aligned} ax + by &\equiv c, \pmod{p} \\ dx + ey &\equiv f, \pmod{q} \end{aligned} \tag{2}$$

can be converted to the system of equations

$$\begin{aligned} ax + by + up &= c, \\ dx + ey + vq &= f, \end{aligned} \tag{3}$$

where u and v must have integral values. The i-solutions of the System of Equations will give the i-solutions of the System of Congruences.

3. Notation. System (1) is called a non-homogeneous system. The corresponding homogeneous system is

$$\sum_{j=1}^{n} a_{ij}x_j = 0, \qquad (i = 1, \cdots, m). \tag{4}$$

Systems (1) and (4) may each be written as a single vector equation, by introducing the vectors:

$$\begin{aligned} A_j &= (a_{1j}, a_{2j}, \cdots, a_{mj}), \qquad (j = 1, \cdots, n), \\ B &= (b_1, b_2, \cdots, b_m), \\ 0 &= (0, 0, \cdots, 0). \end{aligned}$$

Systems (1) and (4), then, become

$$A_1x_1 + A_2x_2 + \cdots + A_nx_n = B. \tag{1'}$$

$$A_1x_1 + A_2x_2 + \cdots + A_nx_n = 0. \tag{4'}$$

4. The complete system. In order to find an e-solution of System (4), m quantities must be made to vanish, namely the left members of the m equations of the system. For an a-solution, it is enough to have them close to 0.

It is also desirable that the variables, x_j, shall be in some way limited in size, as in the case of System A, where severe restrictions on the variables might lead to the ia-solution $(-1, -1)$. Less severe restrictions might lead to a more accurate solution, with larger absolute values of x and y, namely $(-35, -49)$. So there are n further quantities which we would like to have small, namely the values of x_j.

All of the $m+n$ quantities which are to be small are now listed, each with a weight, w_k, with k running from 1 to $m+n$, as follows:

$$\begin{aligned} w_1(x_1 \qquad\qquad\qquad\qquad\quad) &= v_1 \\ w_2(\quad x_2 \qquad\qquad\quad) &= v_2 \\ \cdots\cdots\cdots\

This is called the Complete System.

The values of the weights, w_k, are at our disposal. By decreasing the size of the first n weights, larger values of the variables are possible. In this way, you may choose when formulating the problem whether you prefer an *ia*-solution with small values of the variables and large errors or an *ia*-solution with large values of the variables and smaller errors. You can decide which of the two solutions of System A is better, $(-1, -1)$, or $(-35, -49)$. It is useless to demand the most accurate *ia*-solution, regardless of the size of the variables.

As in the case of Systems (1) and (4), the Complete System, (5) can also be written as a single vector equation:

$$V_1x_1 + V_2x_2 + \cdots + V_nx_n = V, \tag{5'}$$

where the vectors V_j and V are the columns of the extended matrix of System (5). The components of those vectors involve the weights. V is called the deviation vector. The first n components of V are called the solution part, and the last m components are called the error part of V.

DEFINITION. *A lia-solution of a system* (5) *for a given set of weights, w_k, consists of a set of integral values of x_j for which the deviation vector, V, has minimum length.*

As there are at most a finite number of sets of integral values of x_j for which $|V|$ is less than a particular value, one or more of these must minimize $|V|$.

The *lia*-solution of System (5) as defined has the useful property that no other set of integral values of x_j can exist for which the solution part of the deviation vector is shorter without having a longer error part. It will give you the most accurate *ia*-solution of System (4) considering the size of the variables, x_j.

5. The solution-error space, *S*. The vectors $V_1, \cdots, V_n$ are the basis of a space S, called the solution-error space. We shall be concerned with the integral points of S, namely those points obtained by taking integral linear combinations of vectors V_j, such as

$$P = a_1V_1 + a_2V_2 + \cdots + a_nV_n,$$

where $a_1, \cdots, a_n$ are integers. The square length of P is

$$P^2 = P \cdot P = (a_1V_1 + \cdots + a_nV_n)^2, \tag{6}$$

a positive quadratic form in the variables $a_1, \cdots, a_n$.

6. Minkowski on quadratic forms. The problem of finding the minimum value of a positive quadratic form for integral values of the variables has been treated in detail and solved by Minkowski.*

A digest of Minkowski's paper, sufficient for the practical computer, will

* H. Minkowski, Diskontinuitätsberich für arithmetische Äquivalenz, Crelle Journal 129, pp. 220–274, 1905, reprinted in collected works (Gesammelte Abhandlungen) vol. 2, pp. 51–105.

be given. Proofs are omitted. Notations are simplified. Minkowski's paper contains a proof that the proposed procedure will always lead to the smallest value of the quadratic form for integral values of the variables. The example which we work out will suggest the steps of the proof.

7. The method. The shortest non-null integral vector of S is found by a reduction process. We start with $V_1, V_2, \cdots, V_n$, an integral basis of S. At each reduction, one vector, V_c, is culled, after a suitable replacement V_r has been found. To be suitable, V_r must equal V_c plus an integral linear combination of the other vectors of the set, and V_r^2 must be less than V_c^2. When no vector can be culled because of the non-existence of a suitable replacement, the reduction is at an end, and the minimal integral basis of S has been obtained.

It will be shown by examples that in many cases replacements can be found by inspection. Suggestions will be given to aid or improve the inspection process. Nevertheless, when n is 5 or more, the inspection method may fail to uncover replacements which are present. At this point a more systematic procedure will be followed, and this (Minkowski has proved) certainly will find replacements if any exist. It would be uneconomical to use this last procedure when replacements can be found by inspection.

8. Replacements by inspection. In the early stages of a solution, replacements are frequently obvious. It may be noted that V_1+V_2 or V_1-V_2 is shorter than V_1. This happens when the large components of V_1 have the same signs as the corresponding components of V_2, or when they all have opposite signs. Any such obvious relation should be tested.

9. Aided inspection. After culling all the vectors whose replacements can be found by casual inspection, a slightly more systematic procedure must be followed.

Let V be the matrix whose columns are the vectors V_j. Then form the matrix product

$$M = V' \cdot V \tag{7}$$

of V with its transpose, V'.

Any vector of the form

$$V_r = k_1V_1 + k_2V_2 + \cdots + k_nV_n \tag{8}$$

is a suitable replacement for V_c, provided:

1. $k_1, k_2, \cdots, k_m$ are all integers.
2. k_c, the coefficient of the cull, is 1.
3. $V_r^2 < V_c^2$.

With $k_c=1$, V_c can be expressed as an integral linear combination of V_r and the other vectors, so that after the replacement, these will still be an integral linear basis of S.

Condition 3 is somewhat laborious to check in a particular case. V_r^2 is seen to be the value of the quadratic form,

$$\sum M_{ij}k_ik_j.$$

When the values of k_j are limited to 0 and $+1$, the value of this quadratic form is merely the sum of the elements of a certain principal minor of M, namely the minor with rows and columns corresponding to the values of i and j for which k_i and k_j are $+1$. The case when some of the values of k_j are -1 can be handled by changing the corresponding signs in the minor. The illustrations will show how this is done.

It is convenient to speak of a replacement of order 2 or 3 when that replacement is a linear combination of 2 or 3 vectors of the basis. All replacements of order 2 can be found by the following theorem.

THEOREM 1. *$V_r = V_i \pm V_j$ is a suitable replacement for V_i if and only if $\mp 2M_{ij} > M_{jj}$.*

To use Theorem 1 for finding a second order replacement, find M_{ij}, an off-diagonal element of matrix M which in absolute value exceeds one half of M_{jj}, one of the corresponding diagonal elements. If M_{ij} is negative, then $V_r = V_i + V_j$. If M_{ij} is positive, then $V_r = V_i - V_j$ is a replacement for V_i.

Conversely, if no such off-diagonal element exists, then there is no second order replacement for any vector of the set. Every second order replacement can be found by forming the matrix M, and scanning its off-diagonal elements, one by one.

THEOREM 2. *If no suitable second order replacement exists, and $V_r = V_1 + V_2 + V_3$ is a suitable replacement for V_1, then*

(a) *all the off-diagonal elements of the corresponding 3-row principal minor of M must be negative, and*

(b) *their sum in absolute value must exceed $M_{22} + M_{33}$.*

To find a replacement for any vector, V_c, with the aid of any two other vectors of the set, call the vectors V_1, V_2, and V_3. Consider only the 3-row principal minor of M whose elements are $V_1 \cdot V_1$, $V_1 \cdot V_2$, $\cdots$, $V_3 \cdot V_3$. See if the signs of all the off-diagonal elements can be made negative, by changing if necessary the signs of V_2 and V_3. If not, then the replacement is impossible. If the said off-diagonal elements all are negative, then see if their sum satisfies condition (b). This procedure will give every third order replacement.

To discover, if it exists, a replacement for V_1 involving only V_1, V_2, V_3, and V_4, use the following theorem:

THEOREM 3. *If there exists no suitable replacement of the form*

$$V_r = V_1 + k_2V_2 + k_3V_3 + k_4V_4, \tag{9}$$

for V_1, where the values of k_j are limited to -1, $+1$, and 0, then there are none with any integral values of k_j.

In seeking suitable replacements of orders 2, 3, and 4, all 2, 3, and 4 row

principal minors of M must be examined. Theorems 1, 2, and 3 eliminate most of these quickly, so that relatively few remain to be examined quantitatively.

A similar procedure can be used for replacements of order higher than 4, except that in some very rare cases, coefficients, k_j, greater than 1 will have to be used. For this reason, it is very difficult to be sure that all possible replacements have been found, since a thorough search involves such a large number of trials.

10. The final roundup of replacements. After making all the replacements that have been found by inspection, the vectors of the resulting basis of S are arranged in descending order of magnitude, and renamed $\overline{V}_1, \overline{V}_2, \cdots, \overline{V}_n$. To be systematic, we first attempt to find replacements for $\overline{V}_1$; that failing, we try for replacements of $\overline{V}_2$, and so forth. So we must try to find

$$\overline{V}_r = k_1\overline{V}_1 + k_2\overline{V}_2 + \cdots + k_n\overline{V}_n$$

where $k_1 = 1$, all the other k's are integers, and $\overline{V}_r \cdot \overline{V}_r < \overline{V}_1 \cdot \overline{V}_1$. We demand of the method that either it deliver the required vector, or it assure us that no such vector exists.

If no replacement for $\overline{V}_1$ exists, then a similar procedure can be followed to find replacements for $\overline{V}_2$, and so forth. In the end, we shall either have a replacement for one of the vectors, or will know that no replacement exists for any.

From the definition of $\overline{V}_r$, it is seen that $V_r \cdot V_r$ is a quadratic form in k_j. It can be written as a sum of squares, each with a positive coefficient:

$$\begin{aligned} V_r^2 = {} & Y_1(k_n + y_{1,n-1}k_{n-1} + y_{1,n-2}k_{n-2} + \cdots + y_{11}k_1)^2 \\ & + Y_2(k_{n-1} + y_{2,n-2}V_{n-2} + y_{2,n-3}k_{n-3} + \cdots + y_{21}k_1)^2 \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & + Y_{n-1}(k_2 + y_{n-1,1}k_1)^2 \\ & + Y_n(k_1)^2 < V_1^2. \end{aligned} \tag{10}$$

The coefficients, Y_j, are all positive. The value of V_1^2, which is the right member of inequality (10) is a known number, and so are the coefficients, namely the Y's and the y's.

For the purpose of finding a necessary condition on k_r, any of the squared terms of inequality (10) may be dropped. For a sufficient condition, all the terms must be retained.

So, first, retain only the final squared term. This puts limitations on the value of k_1. If $Y_n > V^2$, it requires k_1 to be 0, and no replacement for V_1 exists. Suppose that $k_1 = 1$ is not yet ruled out. Then consider another term of inequality (10), namely the next to last squared term. This limits the value of k_2. Proceeding in this way, every possible integral solution of equation (9) can be found.

To make matters clearer, we give two numerical illustrations. The first is

too simple to justify such a ponderous method, but the reader looking at two unknowns is asked to imagine there are 25 or more.

11. **Example 1.** We shall now solve System A. The homogeneous system is first formed, replacing the constant term by 0. Then the complete system is

$$\begin{aligned} y &= 0 \\ x &= 0 \\ 100(y - 1.41421x) &= 0. \end{aligned}$$

Note the choice of weight, $w=100$, for the last equation. This means, very roughly stated, that we wish the error in the equation, $y-\sqrt{2}x=0$ to be about one hundredth the size of the values of x and y. A choice of $w=10{,}000$ would give a more accurate solution, with larger values of x and y.

Also, in a real problem where there are 25 or so variables, any easily obtained simplification would be justified. So, we could give the first equation, $y=0$, a weight 0. The weight on x would keep both x and y within bounds, since the ratio of x and y would be determined by the last equation.

The vectors forming the basis of the space S are the columns of the complete system, namely

$$\begin{aligned} V_1 &= 1, \quad 0, \quad 100. \\ V_2 &= 0, \quad 1, \quad -141.421. \end{aligned}$$

An obvious replacement is

$$V_3 = V_1 + V_2 = 1, \quad 1, \quad -41.421.$$

V_3 replaces V_2, the longer of V_1 and V_2. The present basis of S is (V_1, V_3). An obvious replacement for V_1 is

$$V_4 = V_1 + V_3 = 2, \quad 1, \quad 58.579.$$

Similarly,

$$\begin{aligned} V_5 &= V_3 + V_4 = 3, \quad 2, \quad 17.158. \\ V_6 &= V_3 + 2V_5 = 7, \quad 5, \quad -7.105. \end{aligned}$$

Let us suppose that no further replacements are obvious. The present basis of S is V_5 and V_6, which we rename $\overline{V}_1$ and $\overline{V}_2$. Then the matrix is

$$V = \begin{Vmatrix} 3, & 2, & 17.158 \\ 7, & 5, & -7.105 \end{Vmatrix}.$$

For M, we have

$$M = V \cdot V' = \begin{Vmatrix} 307.397 & -90.908 \\ -90.908 & 124.481 \end{Vmatrix}.$$

Application of Theorem 1 of Section 9 would immediately direct our attention to the off-diagonal element of M, namely -90.908, which in absolute value exceeds one half of 124.481. This would give us the replacement, V_6+V_6 for the longer of the two vectors, namely V_6. However, we shall overlook this, in order to see how the method of Section 10 operates.

We wish to find, if possible,

$$V_r = \overline{V}_1 + k\overline{V}_2,$$

where k is an integer, and $V_r^2 < \overline{V}_1^2 = 307.397$.

$$\begin{aligned}\overline{V}_r^2 &= (\overline{V}_1 + k\overline{V}_2)^2,\\ &= \overline{V}_1\cdot\overline{V}_1 + 2k\overline{V}_1\cdot\overline{V}_2 + k^2\overline{V}_2\cdot\overline{V}_2,\\ &= 307.397 + 2k(-90.908) + k^2(124.481),\\ &= 124.481(k - 0.73029)^2 + 241.008 < 307.397.\end{aligned}$$

The two members of the inequality are seen to be equal for $k=0$, so it is easily seen to be satisfied for $k=1$.

Following is the suggested arrangement of the computation:

Name	How Obtained	Components			Replaced by
V_1	Given	1	0	100.	V_4
V_2	Given	0	1	−141.421	V_5
V_3	V_1+V_2	1	1	−41.421	V_6
V_4	V_1+V_3	2	1	+58.579	V_6
V_5	V_4+V_3	3	2	+17.158	V_7
V_6	V_3+2V_5	7	5	−7.105	
V_7	V_5+V_6	10	7	+10.053	

Each vector in the final basis of S, namely V_6 and V_7, gives an *ia*-solution of the homogeneous system. V_6, the shortest vector in this basis, gives the *lia*-solution. It is easily seen from the first two components of V_6 that $V_6 = 7V_1+5V_2$, so the said *lia*-solution is $y=7$, $x=5$.

To solve the non-homogeneous complete system, let R be the vector made up of the right hand members of the equations. In the present example, $R=0$, 0, 50, and the system is

$$\begin{aligned} y &= 0\\ x &= 0\end{aligned}$$

$$100y - 141.421x = 50.$$

For the *lia*-solution, we must find the shortest vector, V_s, of the form

$$V_s = -R + k_1V_1 + k_2V_2$$

where k_1 and k_2 are integers. In fact, k_1 and k_2 will be the values of y and x.

As a practical matter, any vector, V_i, from the table of culls can be added to $-R$, since it will be a linear combination of V_1 and V_2. The values of k_1 and k_2 can be read immediately from V_s as its first two components.

In the example, it is noted that $-R+V_4$ has the components 2, 1, 8.579 and so is much shorter than V_7 and V_8. Hence, by inspection, we have the result that

$$V_s = -R + V_4 = -R + 2V_1 + V_3,$$

and the *lia*-solution that $y=2$, $x=1$.

To be absolutely certain that the vector (2, 1, 8.579) is the shortest vector of the prescribed form, we write

$$V_s = k(2,\ 1,\ 8.579) + k_6V_6 + k_7V_7$$

and consider the minimum possible value of $V_s \cdot V_s$ for $k=1$ and integral values of k_6 and k_7. The result is a quadratic form in k, k_6, and k_7. It is written as a sum of squares, as in the solution of the homogeneous case. The result $k_6=k_7=0$ would confirm the result of our inspection. Any other value for k_6 and k_7 would give a better result, one that was overlooked by inspection. The details of the computation are as follows:

$$\begin{aligned} V_s \cdot V_s = {} & 124.481k_6k_6 + 33.573k_6k_7 - 41.954k_6k \\ & + 33.573k_7k_6 + 250.063k_7k_7 + 113.245k_7k \\ & - 41.954kk_6 + 113.245kk_7 + 78.599kk \\ = {} & 11.332k^2 + 241.008(k_7 + 0.51683k)^2 \\ & + 124.481(k_6 + 0.26970k_7 - 0.33703k)^2. \end{aligned}$$

For this to be less than 78.599, with $k=1$, it is easy to see from the squared form that $k_1=-1$ and $k_6=+1$. Therefore,

$$V_s = -R + V_4 + V_6 - V_7 = (-1, -1, -8.579).$$

Therefore, the *lia*-solution of the non-homogeneous system is $y=-1$, $x=-1$.

Following is the suggested arrangement of the computation:

Name	How Obtained	Components		
$-R$	Given	0	0	−50.
V_4	$-R+V_4$	2	1	+8.579.
V_9	$V_6+V_6-V_7$	−1	−1	−8.579.

The solution part of V_9 is the vector with the first two components, which gives the values of y and x. The error part, namely −8.579, gives w times the error, so the latter is −0.08579.

For a solution with a smaller error, larger values of x and y will have to be allowed. For this, change the weight from $w=100$ to $w=10{,}000$, by shifting the decimal point two places in the fifth column of the table of culls, and revise the computation accordingly. This will give the result $y=-49$, $x=-35$, with an error of about .0025. With still larger values of w, solutions with still smaller errors, but larger absolute values of x and y, can be obtained.

For this reason, when attacking a new system, the weights can be assigned arbitrarily, for a first run. Then you can change the weights, as may seem desirable in the light of experience.

12. Example 2. For the second example, we want more than one equation in two variables. The important thing to notice is that the procedure is just the same as in the simpler example.

Solve the system of congruences:

$$\begin{aligned} x_1 \cos 30^\circ + x_2 \sec 40^\circ &= \csc 55^\circ, \pmod 1, \\ x_1 \sin 30^\circ + x_2 \tan 40^\circ &= \cot 55^\circ, \pmod 1. \end{aligned} \tag{11}$$

The corresponding system of equations is:

$$\begin{aligned} x_1 \cos 30^\circ + x_2 \sec 40^\circ + x_3 &= \csc 55^\circ, \\ x_1 \sin 30^\circ + x_2 \tan 40^\circ + x_4 &= \cot 55^\circ. \end{aligned} \tag{12}$$

The homogeneous system is:

$$\begin{aligned} x_1 \cos 30^\circ + x_2 \sec 40^\circ + x_3 &= 0, \\ x_1 \sin 30^\circ + x_2 \tan 40^\circ + x_4 &= 0. \end{aligned} \tag{13}$$

The complete system is:

$$\begin{aligned} 1(x_1 \qquad\qquad\qquad\quad) &= v_1 \\ 1(\quad x_2 \qquad\qquad\quad) &= v_2 \\ 0(\qquad\quad + x_3 \quad) &= v_3 \\ 0(\qquad\qquad\quad + x_4) &= v_4 \\ W(x_1 \cos 30^\circ + x_2 \sec 40^\circ + x_3 \quad) &= v_5 \\ W(x_1 \sin 30^\circ + x_2 \tan 40^\circ \quad + x_4) &= v_6. \end{aligned} \tag{14}$$

Choice of weights 0 for the third and fourth equations is for the purpose of reducing work. Such a possibility was suggested in Example 1.

Arbitrary choice of weight $W=200$ enables one to start the solution. Later W can be adjusted up or down, according as a solution is wanted with a smaller error or with smaller integral values of $x_1, \cdots, x_4$. If you are more concerned in keeping the error small in equation 11a than in equation 11b, the former can be given a larger weight. While the weights are important, their precise values are not.

The vectors forming the basis of space S are, as read from the columns of

the matrix of coefficients of system (14):

$$V_1 = (1,\ 0,\ 0,\ 0,\ W\cos 30°,\ W\sin 30°),$$
$$V_2 = (0,\ 1,\ 0,\ 0,\ W\sec 40°,\ W\tan 40°),$$
$$V_3 = (0,\ 0,\ 0,\ 0,\ W,\ 0\ ),$$
$$V_4 = (0,\ 0,\ 0,\ 0,\ 0,\ W\ ).$$

In the table of culls, each row concerns one vector. The first column gives the number of the vector, the second gives the numbers of the vectors from which it was obtained. The next 6 columns give the six components of each vector. The value shown for each component must still be multiplied by the weight shown at the top of the column.

The next column shows the norm, or squared length of the vector. This takes into account the weights. For example, in the case of V_1, the components are 1, 0, 0, 0, $100\sqrt{3}$, 100. The sum of the squares of these numbers is 40,001. These norms are not necessary for the procedure.

The final column shows which vector finally replaces the vector of that row. V_1 is replaced by V_6, and V_2 by V_5, and so forth. At any point in the formation of this table, one can tell at a glance which vectors have been culled, and which form the current basis of S. The vectors of the final basis of S are seen to be V_{12}, V_{16}, V_{18}, and V_{20}.

TABLE OF CULLS FOR EXAMPLE 2

j	How Obtained	Weights						Norm	Replaced by
		1	1	0	0	200	200		
1		1	0	0	0	0.866025	0.500000	40,001	6
2		0	1	0	0	1.305407	0.839100	96,328	5
3		0	0	1	0	1.000000	0.000000	40,000	7
4		0	0	0	1	0.000000	1.000000	40,000	8
5	2−1	−1	1	0	0	0.439382	0.339100	12,324	11
6	1−2(5)	3	−2	0	0	−0.012739	−0.178200	1,290	14
7	3−5	1	−1	1	0	0.560618	−0.339100	17,173	10
8	4−5	1	−1	0	1	−0.439382	0.660900	25,196	9
9	8+7	2	−2	1	1	0.121236	0.321800	4,738	13
10	7−2(6)	−5	+3	1	0	0.586096	0.017300	13,786	12
11	5−9	−3	3	−1	−1	0.318146	0.017300	891	15
12	10−2(11)	1	−3	3	2	−0.050196	−0.017300	123	
13	9+2(6)	8	−6	1	1	0.095758	−0.034600	515	16
14	6−12	2	1	−3	−2	0.037457	−0.160900	1,097	17
15	11+5(12)	2	−12	14	9	0.067166	−0.069200	520	18
16	13+12	9	−9	4	3	0.045562	−0.051900	353	
17	14−16	−7	10	−7	−5	−0.008105	−0.109000	628	19
18	15−16	−7	−3	10	6	0.021604	−0.017300	89	
19	17−18	0	13	−17	−11	−0.029709	−0.091700	541	20
20	19−12	−1	16	−20	−13	0.020487	−0.074400	495	

All the replacements used in forming the table were found by inspection. Each involved at most the cull and one other vector. This is usual, unless the number of components is large, in which case a few third and fourth order replacements might be needed. These can be found by Theorems 1, 2, and 3 of Section 9. In very rare cases, one replacement might be overlooked, without the procedure of Section 10. The manner of finding the next to the last replacement, V_{19}, by this method will be shown in detail.

Just before finding V_{19}, the current basis of space S is the set (V_{17}, V_{16}, V_{12}, and V_{18}). These have been arranged, for convenience, in order of descending magnitude, and now will be renamed $\overline{V}_1$, $\overline{V}_2$, $\overline{V}_3$, and $\overline{V}_4$. We seek a replacement for the longest, namely $\overline{V}_1$. So, write

$$V_r = k_1\overline{V}_1 + k_2\overline{V}_2 + k_3\overline{V}_3 + k_4\overline{V}_4. \tag{15}$$

To find a suitable replacement for $\overline{V}_1$, we must find a set of integral values for $k_1, \cdots, k_4$, with $k_1 = 1$, and $V_r \cdot V_r < \overline{V}_1 \cdot \overline{V}_1$. But $V_r \cdot V_r$ is seen to be a quadratic form in k_j with matrix $M = V \cdot V'$, where V is the matrix of the vectors $\overline{V}_1$, $\overline{V}_2$, $\overline{V}_3$, $\overline{V}_4$, which computes out in the present case to be

$$M = \begin{Vmatrix} 626.8676 & 58.5128 & 54.7015 & 87.4240 \\ 58.5128 & 352.7802 & -19.5664 & 39.2877 \\ 54.7015 & -19.5664 & 122.7571 & -29.4058 \\ 87.4240 & 39.2877 & -29.4058 & 88.6409 \end{Vmatrix}. \tag{16}$$

Application of Theorem 1 of Section 9 would direct attention to the off-diagonal element 87.4240 of M whose value exceeds half the corresponding diagonal element, 88.6409. Thus, replacement V_{19} would be found. However, we wish to show how the method of Section 10 operates.

So the quadratic form is expressed as a sum of squares:

$$\begin{aligned} \sum M_{ij}k_ik_j = {} & 447.5242k_1^2 + 334.9894(k_2 + 0.304787k_1)^2 \\ & + 113.0032(k_3 - 0.057814k_2 + 0.740719k_1)^2 \\ & + 88.6409(k_4 - 0.33174k_3 + 0.44322k_2 + 0.98627k_1)^2. \end{aligned} \tag{17}$$

We wish to choose k_1, k_2, k_3, k_4 such that

$$\sum M_{ij}k_ik_j < 626.8676. \tag{18}$$

If the coefficient of k_1^2 had turned out to exceed 627, the norm of $\overline{V}_1$, then a mere glance at that term would show that the choice $k_1 = 1$ would contradict inequality (18), and no replacement for $\overline{V}_1$ would be possible. However, the possibility $k_1 = 1$ is not so ruled out in this example.

Now consider the later squared terms (with positive coefficients) one at a time, and in each case choose the new k_j for that term so as to make the new term as small as possible. We try, of course, to keep the sum of the terms less than 626.8676. We know in advance by equation (15) that the values $k_1 = 1$,

$k_2=k_3=k_4=0$ will not satisfy the inequality, for it will make the two members equal. So we find the solution, $k_1=1$, $k_2=0$, $k_3=-1$, $k_4=-1$. These values put into equation (15) would give V_{20}. There is also the solution $k_1=1$, $k_2=k_3=0$, $k_4=-1$, which gives V_{19}. When there are several solutions, it may be a little difficult to make a complete list of all possible ones. When there are no possible solutions, the arithmetic of inequalities will make that fact clear.

The *lia*-solution of the complete homogeneous system can be read from the components of the shortest of the four unculled vectors, namely $V_{18}=-7$, -3, 10, 6, $\cdots$, without weights. The *lia*-solution is $x_1=-7$, $x_2=-3$, $x_3=10$, $x_4=6$. Substitution of these values into the complete homogeneous system gives the successive errors, -7, -3, $10\cdot0$, $6\cdot0$, $(0.021604)(200)$, $(-0.017300)(200)$, the components of V_{18}.

13. Example 3. To save space, only the high points of Example 3 are given. The basis of the space S this time has 5 vectors, so that instead of equation (15), we have

$$V_r = k_1V_1 + k_2V_2 + k_3V_3 + k_4V_4 + k_5V_5, \tag{15'}$$

and for (16), we have, for purposes of illustration,

$$M = \begin{Vmatrix} 26 & -5 & -5 & -5 & -3 \\ -5 & 26 & -5 & -5 & -3 \\ -5 & -5 & 26 & -5 & -3 \\ -5 & -5 & -5 & 26 & -3 \\ -3 & -3 & -3 & -3 & 7 \end{Vmatrix}. \tag{16'}$$

The interesting thing about this example is that Theorems 1, 2, and 3 suggest no replacement for V_1. In fact, trial of the values $+1$, -1, and 0 in every possible combination for the values of $k_1, \cdots, k_5$ in equation (15') fails to give any replacement. If it becomes necessary to try also the values $k_j=+2$ and -2, and if n is 10 or more, then it becomes prohibitive to make an exhaustive search for replacements. So it becomes interesting to see how the method of Section 10 handles the situation. So, write

$$\begin{aligned} \sum M_{ij}k_ik_j = {} & 14.95k_1^2 + 20.43(k_2 - 0.52k_1)^2 \\ & + 23.12(k_3 - 0.34k_2 - 0.34k_1)^2 \\ & + 24.91(k_4 - 0.25k_3 - 0.25k_2 - 2.25k_1)^2 \\ & + 7(k_5 - 0.43k_4 - 0.43k_3 - 0.43k_2 - 0.43k_1)^2. \end{aligned} \tag{17'}$$

For a replacement, we must have

$$\sum M_{ij}k_ik_j < 26. \tag{18'}$$

Taking only the first squared term, we have

$$14.95k_1^2 < 26,$$

so we start with $k_1=1$. In fact for any hope of a replacement for V_1, we must have $k_1=+1$. Next take two squared terms,

$$14.95 + 20.43(k_2 - 0.52)^2 < 26,$$

$k_2=1$ or 0, preferably 1. (If that fails, we can return to the value $k_2=0$.)

Proceeding in this manner, it is found that $k_3=k_4=1$, but $k_5=2$. So we have the desired replacement, $k_r=k_1+k_2+k_3+k_4+2k_5$.

In a more complicated example, with a larger value of n, before you could be sure that no replacements were possible, you may find it necessary to try several different hopeful starts, each of which would prove impossible by the time you considered all the squared terms. But the number of these starts would be finite, and much smaller than the number of trials necessary under the method of inspection of Section 9.

14. The solution of system (13). System (13) is the homogeneous system of Example 2 from which the complete system was formed. If the values $x_1=-7$, $x_2=-3$, $x_3=10$, $x_4=6$, obtained at the end of Section 12, are substituted into System (13), instead of the zeros demanded the results obtained are 0.021604 and -0.017300, namely the last two components of V_{18}. No *ia*-solution with smaller values of x_j (measured by the sum of the squares) can have a smaller error.

15. Solution of system (12). To solve the given non-homogeneous system, (12), put it in the form:

$$(12') \qquad \begin{aligned} -x_0 \csc 55^\circ + x_1 \cos 30^\circ + x_2 \csc 40^\circ + x_3 &= 0, \\ -x_0 \cot 55^\circ + x_1 \sin 30^\circ + x_2 \tan 40^\circ + x_4 &= 0. \end{aligned}$$

For a solution, we must have $x_0=1$. The sum of two solutions would give x_0 the value 2, so the procedure of Section 12 cannot be followed.

Instead, we find a particular *ia*-solution of (12′), namely $x_0=1$, $x_1=x_2=x_3=x_4=0$ (V_{21} below), and to it add a linear combination of *ia*-solutions of (13), in which $x_0=0$. In the result, x_0 would always have the desired value of 1. As

TABULAR SOLUTION OF SYSTEM (12)

i	How Obtained	Weights						
		1	1	1	0	0	200	200
21		1	0	0	0	0	−1.220775	−0.700208
22	21+ 2	1	0	1	0	0	+0.084632	+0.138892
23	22+12	1	1	−2	3	2	+0.034436	+0.121592
24	23+17	1	−6	8	−4	−3	+0.026331	+0.012592
25	24+12	1	−5	5	−1	−1	−0.023865	−0.004708
26	25+18	1	−12	2	9	5	−0.002261	−0.022008

in the tabular solution of system (13), weights are assigned, and the shorter the vector the better the corresponding *ia*-solution. The computation may be put into tabular form.

The shortest vector obtained by inspection is V_{25}. This can be combined with the various vectors in the basis of S in an attempt to find a shorter vector. Among others, this gives V_{26}, which is longer than V_{25}, though its error part is shorter.

To be absolutely sure that you have the *lia*-solution of the non-homogeneous complete system, the matrix can be expressed as a sum of squares, as was illustrated in the case of Example 1.

From V_{25}, the answer to Example 2 is read as $x_1 = -5$, $x_2 = +5$. The values of x_3 and x_4 are each -1. These are the multiples of the modulus which are disregarded when solving a congruence. Substitution of these values of x_1 and x_2 in System (11) leads to errors -0.023865 and -0.005208. V_{26} would give smaller errors, but would use larger values of x_1 and x_2.

MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee

Because of the large number of papers on hand, consideration of new papers for this department has been temporarily suspended.

NOTE ON A QUARTIC CONGRUENCE

L. CARLITZ, Duke University

Known results on the Galois group of the equation

$$x^4 + mx^2 + n = 0 \tag{1}$$

as well as some general theorems about the factorization of a rational prime in an algebraic number field yield information about the congruence

$$x^4 + mx^2 + n \equiv 0 \pmod{p}, \tag{2}$$

where p is an odd prime and m, n are rational integers. It is however easy to obtain such results concerning (2) in a very elementary way as we shall now indicate.

Since p is odd we may assume $m \equiv -2a \pmod{p}$. It is convenient to consider separately two cases: (i) n a square, (ii) n a non-square. Thus in case (i) we consider the congruence

$$x^4 - 2ax^2 + b^2 \equiv 0 \pmod{p} \tag{3}$$

and assume first that $(a^2 - b^2)b^2 \not\equiv 0$. Then none of the numbers

$$2(a-b), \qquad 2(a+b), \qquad a^2 - b^2 \tag{4}$$

is divisible by p; moreover since their product is a square it follows that either all of them are squares or exactly one is a square. We shall prove the following

THEOREM 1. *If* $(a^2-b^2)b^2 \not\equiv 0$ *and each of the numbers* (4) *is a square* (mod p), *then the polynomial* $f(x)=x^4-2ax^2+b^2$ *is congruent to the product of four linear factors, while if only one of the numbers* (4) *is a square then* $f(x)$ *is congruent to the product of two irreducible quadratics.*

To prove the first part of the theorem it suffices to note that if

$$2(a-b) \equiv 4r^2, \qquad 2(a+b) \equiv 4s^2, \qquad a^2-b^2 \equiv 4r^2s^2,$$

then

$$x^4 - 2ax^2 + b^2 \equiv (x+r+s)(x+r-s)(x-r+s)(x-r-s).$$

To prove the second part suppose for example that $2(a-b) \equiv c^2$. Then we may verify that

$$f(x) \equiv (x^2 + cx - b)(x^2 - cx - b). \tag{5}$$

Since $c^2+4b \equiv 2(a+b)$, which by hypothesis is not square, it follows that each of the quadratics in (5) is irreducible (mod p).

The excluded sub-case $(a^2-b^2)b^2 \equiv 0$ is of little interest and obviously yields

$$f(x) \equiv (x^2 - a)^2 \qquad (a^2 \equiv b^2),$$
$$f(x) \equiv x^2(x^2 - 2a) \qquad (b^2 \equiv 0),$$

so that we now have repeated factors.

Turning next to case (ii) we consider the congruence

$$x^4 - 2ax^2 + n \equiv 0 \pmod{p} \tag{6}$$

with n a non-square. It follows that (6) cannot have four roots. We have in this case

THEOREM 2. *Assume* n *not a square. Then the polynomial* $g(x)=x^4-2ax^2+n$ *is irreducible* (mod p) *when* a^2-n *is a non-square. When* a^2-n *is a non-zero square,* $g(x)$ *is congruent to the product of an irreducible quadratic and two distinct linear factors. (The possibilily* $a^2-n \equiv 0$ *is ruled out by the hypothesis concerning* n.)

To prove this theorem suppose first that (6) has a root r. Then

$$g(x) \equiv (x^2 - r^2)(x^2 - n/r^2) \tag{7}$$

and clearly the second factor is irreducible. Moreover since (7) implies $2a \equiv r^2+n/r^2$, we get

$$4(a^2 - n) \equiv \left(r - \frac{n}{r}\right)^2.$$

Conversely when a^2-n is a square we get a factorization

$$g(x) \equiv (x^2 - c)(x^2 - d). \tag{4}$$

Since $cd \equiv n$, one of c, d is a square, the other a non-square and (8) reduces to (7). This takes care of the second part of the theorem. Suppose now that a^2-n is not a square. Clearly $g(x)$ cannot have a linear factor. Also neither the factorization (8) nor

$$g(x) \equiv (x^2 - cx + d)(x^2 + cx + d) \tag{9}$$

can occur, for (8) implies $a^2-n \equiv (c-d)^2$ while (9) implies $n \equiv d^2$.

To give a few numerical examples we cite

$x^4 + 3x^2 + 2 \equiv x^4 - 14x^2 + 2 \equiv (x - 4)(x + 4)(x - 7)(x + 7) \pmod{17}$,

$x^4 - 6x^2 + 4 \equiv (x + 4)(x - 4)(x + 20)(x - 20) \pmod{41}$,

$x^4 + 5x^2 + 1 \equiv x^4 - 6x^2 + 1 \equiv (x^2 + 2x - 1)(x^2 - 2x - 1) \pmod{11}$,

$x^4 - 2x^2 + 2$ irreducible $\pmod{11}$,

$x^4 + 6 \equiv (x - 2)(x + 2)(x^2 - 7) \pmod{11}$,

$x^4 - 2x^2 + 2 \equiv (x - 3)(x + 3)(x^2 - 6) \pmod{13}$,

$x^4 + 13x^2 + 6 \equiv (x - 3)(x + 3)(x^2 + 5) \pmod{17}$,

$x^4 - 4x^2 + 6$ irreducible $\pmod{37}$,

$x^4 - 3x^2 - 1 \equiv (x^2 - 8)^2 \pmod{13}$.

NOTE ON AN ELEMENTARY PROPERTY OF TRIANGLES

GUNNAR STEENSHOLT, University of Oslo

Let P be an arbitrary point within a triangle ABC. Let its distances from A, B and C be R_1, R_2, R_3 ,and let its distances from the sides BC, CA and AB (of lengths respectively a, b, c) be r_1, r_2, r_3. Then we have the theorem of Erdös-Mordell:

$$R_1 + R_2 + R_3 \geqq 2(r_1 + r_2 + r_3).$$

A simple and elementary proof of this theorem is not known (see Toth [1]). However, an elementary theorem of some intrinsic interest, and formally related to the Erdös-Mordell theorem, can be derived very simply as shown below. To the best of the author's knowledge this result does not seem to have been recorded in the literature.

It is plain that for the areas of the triangles involved:

$$ABC - ABP = BCP + CAP,$$

with two other similar equations. If now h_1, h_2, h_3 denote the heights from the vertices A, B, C respectively, and if T is the area of the triangle ABC, then

$$(h_1 - r_1)a + (h_2 - r_2)b + (h_3 - r_3)c = 4T.$$

But now clearly

$$2T = ar_1 + br_2 + cr_3,$$
$$R_1 \geqq h_1 - r_1,$$
$$R_2 \geqq h_2 - r_2,$$
$$R_3 \geqq h_3 - r_3.$$

Therefore

$$aR_1 + bR_2 + cR_3 \geqq 2(ar_1 + br_2 + cr_3),$$

which is the desired result.

If we now introduce the "weighted" means R and r of R_i and r_i respectively $(i=1, 2, 3)$ by putting

$$R = (aR_1 + bR_2 + cR_3)/(a + b + c)$$
$$r = (ar_1 + br_2 + cr_3)/(a + b + c)$$

we obtain

$$R \geqq 2r,$$

which may be regarded as a kind of analogue to the Erdös-Mordell theorem.

If $a=b=c$, we obviously get

$$R_1 + R_2 + R_3 \geqq 2(r_1 + r_2 + r_3).$$

This proves the original Erdös-Mordell theorem for equilateral triangles.

Reference

1. L. Fejes Toth, Lagerungen in der Ebene, auf der Kugel und im Raum, Springer Verlag, Berlin, 1953. See in particular p. 12 and p. 28.

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

Because of the large number of papers on hand, consideration of new papers for this department has been temporarily suspended.

ON THE SEQUENCE FOR EULER'S CONSTANT

S. K. LAKSHMANA RAO, Indian Institute of Science

The convergence of the sequence $a_n = 1 + \frac{1}{2} + \cdots + 1/n - \log n$ as $n \to \infty$ is proved below by expressing a_n as an infinite integral. This procedure has the advantage of leading at once to the integral representation for $\lim a_n = \gamma$ (Euler's Constant) and we may also examine the behaviour of $a_n - \gamma$ for large n.

Noting that $1/r = \int_0^\infty e^{-rz}dx$ $(r>0)$ and $\log n = \int_0^\infty (e^{-z} - e^{-nz})dx/x$ (Frullani's

integral*), we write

$$a_n = \int_0^\infty (e^{-x} + e^{-2x} + \cdots + e^{-nx})dx - \int_0^\infty \frac{e^{-x} - e^{-nx}}{x}\,dx$$
$$= \int_0^\infty \left(\frac{e^{-x} - e^{-(n+1)x}}{1 - e^{-x}} - \frac{e^{-x} - e^{-nx}}{x}\right)dx$$

which can now be split up into a sum of two convergent infinite integrals in the form

$$a_n = \int_0^\infty \left(\frac{e^{-x}}{1 - e^{-x}} - \frac{e^{-x}}{x}\right)dx + \int_0^\infty e^{-nx}\left(\frac{1}{x} - \frac{1}{e^x - 1}\right)dx.$$

The following inequality

$$\frac{1}{2} - \frac{x}{8} < \frac{1}{x} - \frac{1}{e^x - 1} < \frac{1}{2} \qquad (x > 0) \tag{1}$$

can be easily proved by showing that it depends on the elementary inequality $\tanh x < x (x > 0)$. Therefore

$$\int_0^\infty e^{-nx}\left(\frac{1}{x} - \frac{1}{e^x - 1}\right)dx < \int_0^\infty \frac{e^{-nx}}{2}\,dx = \frac{1}{2n}.$$

This proves that a_n converges to a limit γ given by

$$\gamma = \int_0^\infty \left(\frac{e^{-x}}{1 - e^{-x}} - \frac{e^{-x}}{x}\right)dx.$$

Further,

$$a_n - \gamma = \int_0^\infty e^{-nx}\left(\frac{1}{x} - \frac{1}{e^x - 1}\right)dx$$

and by the inequality (1), we have

$$\int_0^\infty e^{-nx}\left(\frac{1}{2} - \frac{x}{8}\right)dx < a_n - \gamma < \int_0^\infty e^{-nx}\cdot\frac{1}{2}\,dx$$

or

$$\frac{1}{2n} - \frac{1}{8n^2} < a_n - \gamma < \frac{1}{2n}.$$

Therefore $a_n - \gamma$ behaves like $1/2n$ and $n(a_n - \gamma) = n(1 + \frac{1}{2} + \cdots + 1/n - \log n - \gamma)$ converges to the limit $\frac{1}{2}$.

* Referee's comment: "Frullani's integral" is a consequence of integration from 1 to n of the previous integral.

A NEW INTERPRETATION OF THE FIELD POSTULATES†

DONALD MILLER, Brookhaven National Laboratory

In this note we shall consider an arithmetical operation $*$ defined as $a * b = a^{\log b}$. The operations $*$ and ordinary multiplication satisfy the field axioms and have certain other interesting properties.

Consider the standard postulates for a field in terms of the quantities $\mathbf{a} \oplus \mathbf{b}$, $\mathbf{a} \otimes \mathbf{b}$, $-\mathbf{a}$, $\mathbf{a}^{-1}$, $\mathbf{0}$, and $\mathbf{1}$. It is easy to verify that the postulates are also satisfied by the following interpretation. Let the positive real numbers be the set of elements, let ab be the interpretation for $\mathbf{a} \oplus \mathbf{b}$, $a * b$ be that for $\mathbf{a} \otimes \mathbf{b}$, the base of the logarithms z (a positive real number, not equal to 1) be the interpretation for $\mathbf{1}$, 1 for $\mathbf{0}$, a^{-1} for $-\mathbf{a}$, and $\bar{a} = z^{1/\log a}$ for $\mathbf{a}^{-1}$.

Suppose one chooses to take ab in its usual role of a multiplication and then takes $a * b$ as a new type of addition. Then an interesting and characteristic property of Boolean algebras is found to hold; namely, the distributivity of addition over multiplication. This property,

$$a * (bc) = (a * b)(a * c)$$

is useful in showing that there is nothing especially unique in numerical systems about the standard distributive law

$$a(b + c) = ab + ac.$$

If $*$ is taken as an addition operation in the interpretation $(K, \cdot, *)$, then there are certain propositions, trivial in the $(K, \cdot, +)$ interpretation, which are no longer trivial since $z \neq 0$. Some examples are

$$a * z^n = a^n, \qquad a * \bar{a}^n = z^n,$$
$$a^{-1} = a * z^{-1}, \qquad az = \bar{a} * az (\neq z).$$

† Research carried out under the auspices of the U. S. Atomic Energy Commission.

THE VOLUME OF A TETRAHEDRON AS A DETERMINANT

S. B. TOWNES, University of Hawaii

In Classroom Notes in the April 1955 MONTHLY, G. M. Petersen gave a simple derivation of the area of a triangle as a determinant in the coordinates of its vertices. It is easy to obtain the volume of a tetrahedron by analogous procedure. In a space of three dimensions a plane area is equal to the square root of the sum of the squares of its projections upon the coordinate planes. It follows that the area of the triangle with the vertices (x_2, y_2, z_2), (x_3, y_3, z_3), and (x_4, y_4, z_4) is, (writing only the first lines of the determinants),

$$\tfrac{1}{2}\sqrt{\begin{vmatrix} x_2 & y_2 & 1 \end{vmatrix}^2 + \begin{vmatrix} y_2 & z_2 & 1 \end{vmatrix}^2 + \begin{vmatrix} z_2 & x_2 & 1 \end{vmatrix}^2}.$$

The equation of the plane through these same points is

$$\begin{vmatrix} x & y & z & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0.$$

Consider the tetrahedron with the vertices (x_i, y_i, z_i), $i=1, 2, 3, 4$. Using the normal form of the preceding equation, one gets the altitude through (x_1, y_1, z_1) in the form

$$\pm \frac{1}{\sqrt{\begin{vmatrix} x_2 & y_2 & 1 \end{vmatrix}^2 + \begin{vmatrix} y_2 & z_2 & 1 \end{vmatrix}^2 + \begin{vmatrix} z_2 & x_2 & 1 \end{vmatrix}^2}} \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}.$$

From this it readily follows that the volume of the tetrahedron is

$$\pm \frac{1}{3!} \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}.$$

By mathematical induction the content of a simplex in n dimensions can be found by the same method.

THE EVALUATION OF TWO INFINITE INTEGRALS

C. D. Olds, San Jose State College

The "differential equation" approach to the evaluation of definite integrals has great student appeal, and the advanced calculus teacher should have a few examples tucked away in his notes. For example, consider the integral

$$u \equiv \int_0^\infty \frac{a \cos mx}{a^2 + x^2}\, dx, \tag{1}$$

where, in order to make the discussion as brief as possible, we shall assume that $a>0$ and that $m>0$. Using double integration by parts, we see at once that

$$u = -\frac{1}{m^2} \int_0^\infty \cos mx \frac{d^2}{dx^2}\left(\frac{a}{a^2 + x^2}\right) dx. \tag{2}$$

But

$$\frac{d^2}{dx^2}\left(\frac{a}{a^2 + x^2}\right) = -\frac{d^2}{da^2}\left(\frac{a}{a^2 + x^2}\right),$$

and hence (2) shows that u satisfies the differential equation

$$\frac{d^2u}{da^2} = m^2u,$$

whose solution is

$$u = Ae^{ma} + Be^{-ma}, \tag{3}$$

where A and B are functions of m and not of a. However, if in (1) we let $x=ay$ we find that

$$u = \int_0^\infty \frac{\cos may}{1+y^2}\,dy, \tag{4}$$

which shows that u is a function of ma; hence the constants A and B in (3) are independent of m. Letting, then, $m\to\infty$ in (3), and using the fact that (4) converges, we deduce that $A=0$. Likewise letting $m\to 0$ we conclude that $B=\pi/2$. Thus

$$\int_0^\infty \frac{a\cos mx}{a^2+x^2}\,dx = \frac{1}{2}\pi e^{-ma}.$$

Now if

$$v \equiv \int_0^\infty \frac{a\sin mx}{x(a^2+x^2)}\,dx,$$

then

$$\frac{dv}{dm} = u = \frac{1}{2}\pi e^{-ma}$$

so that $v=\pi(1-e^{-ma})/2a$. Hence

$$\int_0^\infty \frac{a^2\sin mx}{x(a^2+x^2)}\,dx = \frac{1}{2}\pi(1-e^{-ma}). \tag{5}$$

Differentiating (5) twice with respect to m, we find that

$$\int_0^\infty \frac{x\sin mx}{a^2+x^2}\,dx = \frac{1}{2}\pi e^{-ma}. \tag{6}$$

Adding (5) and (6) we obtain the evaluation

$$\int_0^\infty \frac{\sin mx}{x}\,dx = \frac{1}{2}\pi.$$

It is clear, of course, that all the operations involved can be established on the basis of uniformity.

A CONTINUATION PROBLEM

C. P. GADSDEN, Tulane University

If one solution y_1 of $y''+f_1y'+f_2y=0$ is known, a second (linearly independent) solution is given by any solution of the reduced equation

$$y_1y' - y_1'y = \exp\int_{x_0}^{x} -f_1dx = w(x). \tag{1}$$

Thus $y_2=y_1\int_{x_0}^{x}(w/y_1^2)dx$. Clearly a problem arises when y_1 has zeros. Suppose $y_1(x_1)=0$ (hence $y_1'(x_1)\neq 0$, since we assume of course $y_1\not\equiv 0$). Choose x_2 and x_3 so that x_1 is the only zero of y_1 in the interval $x_2<x_3$, and consider a third solution $y_3=y_1\int_{x_3}^{x}(w/y_1^2)dx$. From (1) we have, by continuity at x_1, $y_3(x_1)=y_2(x_1)=-w(x_1)/y_1'(x_1)$. To compare the derivatives of y_2 and y_3 at x_1 we integrate the easily verified relation $d(y'/w)=(-f_2y/w)dx$, first between x_1 and x_2 for $y=y_2$, and then between x_1 and x_3 for $y=y_3$; thus $y_2'(x_1)=w(x_1)\left[1/y_1(x_2)-\int_{x_2}^{x_1}(f_2y_2/w)dx\right]$ and $y_3'(x_1)=w(x_1)\left[1/y_1(x_3)-\int_{x_3}^{x_1}(f_2y_3/w)dx\right]$, so that generally $y_2'(x_1)\neq y_3'(x_1)$, and y_3 is in general not the continuation of y_2 across x_1. The continuation is readily obtained, however, in the form $y_2=Ky_3+Cy_1$, since y_3 and y_1 form a fundamental set of solutions. The constants are easily evaluated as $K=y_2(x_1)/y_3(x_1)=1$ and $C=\left[y_2'(x_1)-y_3'(x_1)\right]/y_1'(x_1)$, which can be computed from the formulas obtained above. Continuation of y_2 across further zeros of y_1 is carried out interval by interval using the same method and basic relations.

Reference

Witold Hurewicz, Ordinary differential equations in the real domain, Brown University, 1943, pp. 45–47.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1231. *Proposed by A. W. Walker, University of Toronto*

Show that $x^3+y^3+z^3-3xyz=a^3$ is a surface of revolution.

E 1232. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Arbitrary parallel lines drawn through the vertices A, B, C of a triangle

intersect the circumcircle in A', B', C'. Show that A'', B'', C'', the symmetrics of these points with respect to the midpoints of BC, CA, AB, respectively, lie on a line perpendicular to the parallel lines and passing through a fixed point of the triangle.

E 1233. *Proposed by Joseph Andrushkiw, Seton Hall University*

Denote the sides and inradius of a triangle by a_0, b_0, c_0, and r_0. The points of contact form a new triangle whose sides and inradius are a_1, b_1, c_1, and r_1. Repeating the process one obtains the sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{r_n\}$. Show that as $n \to \infty$

$$\lim r_n/a_n = \lim r_n/b_n = \lim r_n/c_n = \sqrt{3}/6.$$

E 1234. *Proposed by Leo Moser and J. R. Pounder, University of Alberta*

(a) Find 3 distinct positive integers, relatively prime in pairs, such that the sum of any two is divisible by the third. Prove that these integers are unique.

(b) Find 3 distinct positive integers such that the product of any two leaves a remainder of 1 on division by the third. Prove that the numbers are unique.

E 1235. *Proposed by D. S. Greenstein, University of Pennsylvania*

Let $f_0(x)$ be bounded and integrable over $a \leqq x \leqq b$, and let

$$f_n(x) = \int_a^x f_{n-1}(t)dt, \qquad n = 1, 2, 3, \cdots, \ a \leqq x \leqq b.$$

Evaluate

$$\sum_{n=1}^{\infty} f_n(x).$$

SOLUTIONS

Maximum Cross Section of Unit Cube

E 1201 [1956, 120]. *Proposed by C. S. Ogilvy, Hamilton College*

What is the area of the maximum cross section of the unit cube?

Solution by A. R. Hyde, West Hartford, Conn. Equatorial sections of the circumscribed sphere cut the cube in the following possible configurations:

(a) section includes 4 vertices of the cube (2 diametrically opposed pairs) and separates the others, 2 on each side of the plane; configuration a rectangle $\sqrt{2}$ by 1 with area $\sqrt{2}$;

(b) section includes 2 vertices (diametrically opposed) and separates the others, 3 on each side of the plane; configuration a parallelogram ranging in area from a rhombus with diagonals $\sqrt{3}$ and $\sqrt{2}$ and area $\sqrt{3/2}$ (plane bisecting 2 edges of the cube) to the rectangle of (a) as a limiting case;

(c) section separates vertices into two coplanar groups of 4; configuration

a parallelogram ranging from a unit square (plane parallel to a surface of the cube) to the rectangle of (a) as a limiting case;

(d) section separates vertices into two noncoplanar groups of 4; configuration a hexagon with pairs of equal opposite sides and equal opposite angles, and whose limits are the rhombus of (b) and the rectangle of (a).

It can be shown that planes parallel to any of these equatorial planes cut off successively smaller areas (or, at most, no greater areas) as the latitude increases. In cases (a), (b), and (c) the equatorial section is the same as if made on a unit-square prism of indefinite length. Parallel sections of this prism are congruent, whereas in cases (a) and (b) the areas decrease with increasing latitude because of the bounding surface, and in case (c) they are constant until a vertex is reached, then decrease. The hexagon of (d) may be considered as the parallelogramic section of the indefinitely long prism diminished by congruent triangles at two opposite vertices. For parallel planes these triangles remain similar as the base of one moves toward the vertex, the base of the other away from the vertex, at the same rate. Thus the sum of the triangles increases and the hexagon correspondingly decreases until a vertex is reached and the sectional area becomes pentagonal; thereafter, case (d) is like the others.

Hence the greatest section is that of one of the equatorial planes, namely that of case (a), the $\sqrt{2}$ by 1 rectangle with area $\sqrt{2}$.

Also solved by Leon Bankoff, M. S. Klamkin, and Azriel Rosenfeld.

Klamkin gave an analytical solution and remarked that it would be more difficult to determine the maximum cross section of the higher ordered regular polyhedra. The other two solutions were intuitive answers.

Editorial Note. Since the maximum section of a convex body possessing a center of symmetry must pass through the center of symmetry, one needs to consider only those sections that pass through the center of the cube.

Property of a General Simplex

E 1202 [1956, 120]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let O be an arbitrary point on an arbitrary line λ passing through the centroid G of a tetrahedron $ABCD$. If λ cuts the planes BCD, CDA, DAB, ABC in A', B', C', D', show that

$$A'O/A'G + B'O/B'G + C'O/C'G + D'O/D'G = 4.$$

Solution by Azriel Rosenfeld, Columbia University. Clearly

$$A'O/A'G = OBCD/GBCD = 4\,OBCD/ABCD,$$

with similar expressions for $B'O/B'G$, $C'O/C'G$, and $D'O/D'G$. Therefore

$$A'O/A'G + B'O/B'G + C'O/C'G + D'O/D'G$$
$$= 4(OBCD + OCDA + ODAB + OABC)/ABCD = 4.$$

Also solved by Hüseyin Demir, A. R. Hyde, P. W. M. John, M. S. Klamkin, Josef Langr, Jože Ulčar, Chih-yi Wang, and the proposer.

Many of the solutions were analytical. N. A. Court pointed out that the problem is stated and solved in *National Mathematics Magazine*, vol. 15, no. 6, March 1941, p. 275, art. 9b. It is stated there that the analogous proposition for (affine) two dimensional space is also true. Since the corresponding proposition for affine one dimensional space is easily established, one might conjecture that the proposition holds for the general simplex in affine n-dimensional space. Such is the case; Ulčar furnished an analytical proof, and Rosenfeld's synthetic proof is easily generalized.

Klamkin pointed out that the given problem is a special case of the following problem found in Altshiller-Court, *Modern Pure Geometry*, p. 25, Ex. 38: "The areas of the faces of a given polyhedron are inversely as the perpendiculars from a point O, and OO' meets these planes in $P_1, P_2, P_3, \cdots, P_n$, respectively; prove that

$$O'P_1/OP_1 + O'P_2/OP_2 + \cdots + O'P_n/OP_n = n."$$

Taking $n=4$ and replacing O' by O and O by G, we obtain the desired result.

An Extension of Poretsky's Law

E 1203 [1956, 120]. *Proposed by S. I. Birnbaum, Polytechnic Institute of Brooklyn*

Prove that for given sets X, Y, T we have $X=Y=0$ if and only if

$$T = (X \cap T') \cup (Y \cap T') \cup (X' \cap T \cap Y').$$

I. *Solution by C. F. Pinzka, Educational Testing Service, Princeton, N. J.*

Necessity: If X is non-empty, it contains an element a. If a is in T, it is not in $X \cap T'$, $Y \cap T'$, $X' \cap T \cap Y'$, nor in their union T. Assuming a is in T' also leads to a contradiction. Hence $X=0$. Similarly, $Y=0$.

Sufficiency: Substituting $X=Y=0$ in the given expression yields $T=T$.

II. *Solution by Lawrence Glasser, University of Wisconsin.*

Necessity: $(X \cap T') \subset T$, whence $X=0$; $(Y \cap T') \subset T$, whence $Y=0$.

Sufficiency: $(0 \cap T') \cup (0 \cap T') \cup (1 \cap T \cap 1) = T$.

III. *Solution by R. M. Conkling, New Mexico College of Agriculture and Mechanic Arts.* Factoring T' from the first two summands, we have

$$T = [T' \cap (X \cup Y)] \cup [T \cap (X' \cup Y')].$$

The right side is the symmetric difference of T and $X \cup Y$, *i.e.*, the set of elements in either T or $X \cup Y$, but not in both. This symmetric difference is T if and only if $X \cup Y = 0$, or, equivalently, if and only if X and Y are both null.

IV. *Solution by Roy Dubisch, Fresno State College.* We use the equivalence between the algebra of the class of all subsets of any set and the algebra of statements to rewrite the given equality as:

t equivalent to $(x \wedge \sim t) \vee (y \wedge \sim t) \vee (\sim x \wedge t \wedge \sim y)$.

Suppose t is true. Then $\sim t$ is false and $x \wedge \sim t$ and $y \wedge \sim t$ are false. Therefore the right hand side is true if and only if $\sim x \wedge \sim y$ is true or, hence, if and only if x and y are both false. Suppose now that t is false. Then $\sim x \wedge t \wedge \sim y$ is false and both $x \wedge \sim t$ and $y \wedge \sim t$ must be false. Since $\sim t$ is true this occurs if and only if both x and y are false.

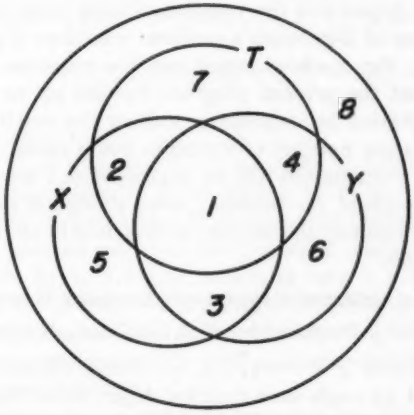
V. *Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn.* The condition $X = Y = 0$ is obviously sufficient. To prove the necessity, consider the Venn diagram. We have

$$(1) + (2) + (4) = (3) + (5) + (6);$$

whence

$$(1) = (2) = (4) = (3) = (5) = (6) = 0,$$

and $X = Y = 0$.



VI. *Solution by E. D. Schell, Sperry Rand Corporation.* By Poretsky's law (see, for example, Birkhoff and MacLane, *Survey of Modern Algebra*, p. 317), for any given A and T, $A = 0$ if and only if

$$T = (A \cap T') \cup (A' \cap T).$$

Now set $A = X \cup Y$ and we have

$$T = (X \cap T') \cup (Y \cap T') \cup (X' \cap T \cap Y')$$

if and only if $X \cup Y = 0$, or $X = Y = 0$.

Also solved by J. L. Alperin, W. A. Al-Salam, Howard Banilower, G. E. Bardwell, Zamir Bavel, W. A. Cava, Peter Crawley, A. E. Danese, I. A. Dodes, David Ellis, A. J. Goldman, D. S. Greenstein, A. S. Gregory, Vern Hoggatt, A. R. Hyde, Lawrence Isenecker, P. W. M. John, K. A. MacDonald, D. C. B. Marsh, R. G. McDermot, J. B. Muskat, E. L. Nath, Donald Norris, C. S.

Ogilvy, J. K. Patterson, J. D. Riley, L. A. Ringenberg, Stewart Robinson, Azriel Rosenfeld, David Rothman, J. P. Scholz, A. J. Tingley, Chih-yi Wang, David Zeitlin, and the proposer.

Schell included the following interesting note with his solution:

"Mr. Richard Ash, of the Programming Research Group of Remington Rand Univac Division, has devised a set of machine instructions which enables one to settle the validity of propositions in symbolic logic on the Univac, which is a large scale general purpose electronic computer. Because of the ease in stating Birnbaum's problem as a proposition in symbolic logic, I asked Mr. Ash to try it out using his machine instructions, which he kindly undertook to do. The problem was translated into symbolic logic form using the Polish parentheses-free notation, which is the convention adopted for the machine instructions. After the instructions were read into the machine, it typed out the query, 'How many variables?' For this problem the response '3' was typed in. The machine then typed out the request, 'Please type in problem.' The appropriate translation of Birnbaum's problem was then typed in. In something less than a second, the machine typed out the response, 'Result: True.' Mr. Ash informs me that the present program handles up to ten variables, which is an arbitrary limitation he imposed in writing the machine instructions, and an extension to a larger number of variables could readily be made.

"I believe your contributors will be impressed as I was with the speed with which the machine solved the problem, even though it pursued what appears to be a round about manner; namely an examination of all possible combinations of the truth variables."

An Application of Fundamental Continuity Theorems

E 1204 [1956, 120]. *Proposed by A. J. Goldman, Princeton University*

Let C be the circle $x^2+y^2=1$, $f(x, y)$ a continuous real-valued function defined on C, and A an angle such that $0<A\leqq\pi$. Show that there are two radii of C which form an angle A and have endpoints at which $f(x, y)$ has the same value.

Solution by L. R. Ford, Illinois Institute of Technology. Put $f(x, y)=F(\theta)$, where θ is the central angle. The periodic function $F(\theta)-F(\theta+A)\geqq 0$ at the maximum of $F(\theta)$ and $\leqq 0$ at the minimum; hence $=0$ somewhere.

Also solved by Louis Brickman, C. N. Campopiano, Peter Crawley, David Ellis, David Freedman, D. S. Greenstein, H. J. Hamilton, Vern Hoggatt, A. R. Hyde, M. S. Klamkin, Joseph Lehner, D. C. B. Marsh, J. C. Mathews, C. S. Ogilvy, C. F. Pinzka, J. D. Riley, L. A. Ringenberg, Azriel Rosenfeld, J. W. Ross, David Rothman, J. A. Tierney, Jože Ulčar, Chih-yi Wang, J. V. Whittaker, David Zeitlin, and the proposer.

Special Case of an Inequality Due to Borg

E 1205 [1956, 121]. *Proposed by H. J. Cohen, City College of New York*

If $f''(x)$ is continuous on $[0, 1]$ and if $f(0)=f(1)=0$, show that

$$\int_0^1 |f''(x)/f(x)|\,dx > 4.$$

Solution by Martinus Esser, University of Maryland. There exist, by Rolle's Theorem, a point (X, Y) at which $|f(x)|$ is maximum, and, by the Mean Value Theorem, two abscissas a, b such that

$$0 < a < X < b < 1, \qquad f'(a) = Y/X, \qquad f'(b) = -Y/(1-X).$$

Then

$$\int_0^1 |f''(x)/f(x)|\,dx > \int_0^1 |f''(x)/Y|\,dx > \left|\int_a^b f''(x)dx\right| \Big/ |Y|$$
$$= |f'(b) - f'(a)| / |Y| = 1/X(1-X) \geq 4.$$

The given inequality is the best possible because for $f(x) = x$ on $[0, 1/2-\epsilon]$, $f(x) = 1-x$ on $[1/2+\epsilon, 1]$ and $f''(x) \leqq 0$ on $[0, 1]$ we have

$$\int_0^1 |f''(x)/f(x)|\,dx < 4/(1-2\epsilon),$$

which is arbitrarily close to 4.

Also solved by Paul Chessin, N. J. Fine, A. J. Goldman, H. Kestelman, M. S. Klamkin, Viktors Linis, T. F. Mulcrone, D. J. Newman, John Selfridge, R. E. Shafer, Chih-yi Wang, Albert Wilansky, and David Zeitlin.

Several solvers pointed out that the proposed inequality is a special case of an inequality due to A. Beurling, which in turn is a special case of an inequality due to G. Borg. See this MONTHLY [1956, 27]. Chessin pointed out that in the *Quarterly Journal of Applied Mathematics*, Oct. 1955, p. 335, is a reference to the proposed inequality with an indication that it is due to Liapounoff and is the best estimate.

It should be noted that $f(x) \equiv 0$ is to be ruled out, since in this case the integral does not exist.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4703. *Proposed by R. E. Bellman, Rand Corporation*

Show that the conditions (a) $u_n \geqq u_{n-1} - a_{n-1}$, $a_n \geqq 0$, $\sum_{n=0}^{\infty} a_n < \infty$, (b) $u_n \leqq M$, imply the convergence of the sequence $\{u_n\}$.

4704. *Proposed by L. E. Clarke, University College of the Gold Coast*

If m is a positive integer greater than 1, show that a necessary and sufficient condition that m be prime is that, for every integer n,

$$\binom{n}{m} \equiv [n/m] \pmod{m}.$$

4705. *Proposed by Albert Wilansky, Lehigh University*

It is possible to construct an additive but discontinuous real function of one real variable. Is it possible to construct one with the property that $f(x) = f(y)$ implies $x = y$? (This would give an algebraic isomorphism of the additive group of the reals which is not continuous.)

4706. *Proposed by S. W. Golomb, Cambridge, Mass.*

If $f(x)$ and $g(x)$ are real periodic functions of the real variable x, having incommensurable periods, is it possible for $f(x)+g(x)$ to be periodic?

4707. *Proposed by Charles Fox, McGill University*

A magic matrix is one whose elements are the numbers of a magic square, *i.e.*, every row, column and diagonal has the same sum. (a) Show that a 3 by 3 magic matrix inverts into a magic matrix. (b) Can this result be extended to magic matrices of higher order?

SOLUTIONS

A Summation of Cosines

4636 [1955, 259]. *Proposed by Emma Lehmer, Berkeley, California*

Show that, if n is a prime of the form $8m+5$, then

$$\sum_{\nu=0}^{n-1} \cos \frac{2\pi\nu^4}{n} = \sqrt{n}.$$

For what other values of n is this true?

Solution by Leonard Carlitz, Duke University. A. Let p be a prime $\equiv 5 \pmod 8$ and let r be a primitive root (mod p). Put

$$\eta_k = \sum_{s=0}^{m-1} \epsilon^{r^{4s+k}} \qquad (k = 0, 1, 2, 3)$$

where $p = 4m+1$ and $\epsilon = e^{2\pi i/p} = \cos(2\pi/p) + i \sin(2\pi/p)$. Next define the Lagrange resolvents (see, *e.g.*, Bachmann, *Die Lehre von der Kreisteilung*, p. 168)

$$(i, \epsilon) = \eta_0 + i\eta_1 - \eta_2 - i\eta_3,$$
$$(-i, \epsilon) = \eta_0 - i\eta_1 - \eta_2 + i\eta_3.$$

We have also

$$\eta_0 + \eta_1 + \eta_2 + \eta_3 = -1,$$
$$\eta_0 - \eta_1 + \eta_2 - \eta_3 = \sqrt{p},$$

the last from the evaluation of a Gauss sum. It follows that

$$1 + 4\eta_0 = (i, \epsilon) + (-i, \epsilon) + \sqrt{p},$$
$$1 + 4\eta_2 = -(i, \epsilon) - (-i, \epsilon) + \sqrt{p}.$$

Noting that η_2 and η_0 are conjugates, and that $1^4, 2^4, \cdots, (p-1)^4$ give the set of biquadratic residues counted four times, we get

$$\sum_{s=0}^{p-1} \cos \frac{2\pi s^4}{p} = 1 + 2(\eta_0 + \eta_2) = \sqrt{p}, \tag{1}$$

the stated result.

It also follows from the above that

$$1 + 2\eta_1 + 2\eta_3 = -\sqrt{p},$$

so that

$$\sum_{s=0}^{p-1} \cos \frac{2\pi k s^4}{p} = -\sqrt{p}, \tag{2}$$

where k is any quadratic non-residue of p. We may combine (1) and (2) in the form

$$\sum_{s=0}^{p-1} \cos \frac{2\pi a s^4}{p} = \left(\frac{a}{p}\right)\sqrt{p}, \qquad (a \not\equiv 0). \tag{3}$$

For $p \equiv 1 \pmod 8$, the numbers $1+4\eta_0$, $1+4\eta_2$ are determined by means of

$$\sqrt{p} + \sqrt{2(p + a\sqrt{p})}, \tag{4}$$

where $p=a^2+b^2$, $a\equiv -1 \pmod 4$, (see Fricke, *Lehrbuch der Algebra*, v. 1, pp. 441–443), but it is not known which value of (4) corresponds to $1+4\eta_0$. Since for $p\equiv 1 \pmod 8$,

$$1 + 4\eta_0 = \sum_{s=0}^{p-1} \cos \frac{2\pi s^4}{p},$$

it is clear from (4) that (1) does not hold.

B. To find values of n such that

$$\sum_{s=0}^{n-1} \cos \frac{2\pi s^4}{n} = \sqrt{n}, \tag{5}$$

we make use of some known results concerning

$$S_\rho = \sum_{s=0}^{n-1} \rho^{s^4},$$

where ρ is a primitive nth root of unity (see Landau, *Vorlesungen über Zahlentheorie*, v. 1, Th. 281, 307, 312):

(a) $$S_{\rho_1\rho_2} = S_{\rho_1}S_{\rho_2},$$

where ρ_1 is an n_1th root and ρ_2 an n_2th root of unity, $(n_1, n_2)=1$,

(b) $$S_\rho = p^{l-1} \qquad (n = p^l,\ p > 2,\ 2 \leqq l \leqq 4),$$

(c) $$S_\rho = p^3 S_{\rho^{p^4}} \qquad (n = p^l,\ l > 4).$$

First, if $n=m_1^2$, where m_1 is quadratfrei and odd, it follows from (a) and (b) that $S_\rho = m_1$.

Second, if $m_2 = p_1 \cdots p_{2r}$ where the p's are distinct primes $\equiv 3 \pmod 4$, then every square (mod m) is necessarily congruent to a fourth power, so that

$$S_\rho = \sum_{s=0}^{m_2-1} \rho^{s^2} = \left(\frac{k}{m_2}\right)\sqrt{m_2} \qquad (\rho = e^{-2\pi k i/m}),$$

Third, for $m_3=4$, we have $S_i = 1+i$, $S_{-i} = 1-i$.

Fourth, for $m_4 = p \equiv 5 \pmod 8$ we have the case treated in A above.

It follows at once that (5) holds in each of these four cases. Moreover we may combine by means of (a) to get the following possibilities:

$$\begin{aligned} n &= m_1^{2e_1} m_2^{e_3} m_3^{e_2}, \\ n &= m_1^{2e_1} m_2^{e_2} m_4^{e_4}, \end{aligned} \tag{6}$$

where each $e_j = 0$, 1 and $(m_1, m_2) = (m_1, m_3) = 1$. For example, when $n = m_3 m_4 = m_2 p$, let $up + vm_2 = 1$, $\rho = e^{2\pi i/n}$, $\rho_2 = e^{2\pi i u/m_2}$, $\rho_4 = e^{2\pi i v/p}$. Then

$$S = S_2 S_4 = \left(\frac{u}{m_2}\right)\sqrt{m} S_4,$$

$$\sum_{s=0}^{n-1} \cos \frac{2\pi s^4}{n} = \left(\frac{u}{m_2}\right)\sqrt{m_2}\left(\frac{y}{p}\right)\sqrt{p} = \left(\frac{u}{m_2}\right)\left(\frac{v}{p}\right)\sqrt{m_2 p}.$$

But since $up \equiv 1 \pmod{m_2}$ and $vm_2 \equiv 1 \pmod{p}$,

$$\left(\frac{u}{m_2}\right)\left(\frac{v}{p}\right) = \left(\frac{p}{m_2}\right)\left(\frac{m_2}{p}\right) = +1,$$

by the quadratic reciprocity theorem. Hence (5) holds in this case.

Presumably (6) enumerates all the cases in which (5) holds.

Also solved (part A only) by J. H. Hodges, R. E. Shafer, and the Proposer.

Integral Points on a Sphere

4643 [1955, 446]. *Proposed by Norman Anning, Alhambra, California*

If r is a prime of the form $4k+1$ or is the product of such primes, show that the number of points with integral coordinates on the sphere

$$x^2 + y^2 + z^2 = r^2$$

is $6r$.

Remarks by C. D. Olds, San Jose State College. In a letter written in 1884 to Ch. Hermite, T. J. Stieltjes proved by means of elliptic functions that if $n = p^k$, $p \equiv 1 \pmod 8$, p prime, then $N_3(n^2) = 6p^k$, where $N_3(n^2)$ denotes the number of representations of n^2 as the sum of three squares. ["Lettre 45, *Correspondence d'Hermite et de Stieltjes*, vol. 1, Paris, 1905, pp. 89–94.] In 1907 this formula was generalized by A. Hurwitz [*Mathematische Werke*, vol. 2, Basel, 1933, p. 751.]

The general formula is as follows: Let

$$n = 2^k m = 2^k PQ, \quad P = \prod_{r=1}^{r} p_r^{a_r}, \quad Q = \prod_{r=1}^{s} q_r^{b_r},$$

where each p_r is a prime $\equiv 1 \pmod 4$, and each q_r is a prime $\equiv 3 \pmod 4$, then

$$N_3(n^2) = 6P \prod_{r=1}^{s}\left[q_r^{b_r} + 2\,\frac{q_r^{b_r} - 1}{q_r - 1}\right].$$

Arithmetical proofs have been given by G. Pall [*Journal of the London Mathematical Society*, vol. 5 (1930), pp. 102–105], and by C. D. Olds [*Bulletin of the American Mathematical Society*, vol. 47 (1941), pp. 499–503.]

Similar discussions were submitted by Leonard Carlitz, M. S. Klamkin, A. Walfisz, and Chih-yi Wang.

Isomorphic Groups

4644 [1955, 447]. *Proposed by Azriel Rosenfeld, Columbia University*

Prove that in no field K can the additive group K^+ be isomorphic to the

multiplicative group K^*.

Solution by R. E. Priest, Bell Telephone Laboratories, New York City. If the characteristic of K is not equal to 2, then K^+ has precisely one element of order 2, *viz.* 0, and K^* has precisely two elements of order 2, *viz.* 1 and -1.

If the characteristic of K equals 2, then all elements of K^+ are of order 2 while K^* has precisely one element of order 2, *viz.* $1 = -1$.

In no case, therefore, can K^+ and K^* be one to one.

Also solved by W. J. Blundon, G. U. Brauer, Leonard Carlitz, R. C. Courter, Helen F. Cullen, M. P. Drazin, David Ellis, M. P. Epstein, V. D. Gokhale, A. J. Goldman, William Googe, R. D. Gordon, C. D. Gorman, D. S. Greenstein, B. Gross, A. S. Hendler, J. Horváth, H. H. Johnson, Tung-Po Lin, D. C. B. Marsh, E. C. Milner, T. O. Moore, F. D. Parker, G. B. Robison, H. D. Ruderman, J. J. Schäffer, and W. R. Scott.

A Determinant Divisible by a Prime

4645 [1955, 447]. *Proposed by Leonard Carlitz, Duke University*

Let $0 \leqq r \leqq p-1$, where p is prime, and let $a_0, a_1, \cdots, a_r$ be integers not divisible by p. Define the determinant of order $r+1$

$$\Delta_r = \left| a_i^{p^j} \right| \qquad (i, j = 0, 1, \cdots, r).$$

Show that

$$\Delta_r \equiv 0 \pmod{p^{r(r+1)(r+2)/6}}.$$

Solution by the Proposer. We shall prove the following stronger result. Let $m \geqq 0$ and define

$$\Delta_r^{(m)} = \left| a_i^{p^{m+j}} \right| \qquad (i, j = 0, 1, \cdots, r).$$

Then we show that

$$\Delta_r^{(m)} \equiv 0 \pmod{p^e}, \tag{1}$$

where $e = mr(r+1)/2 + r(r+1)(r+2)/6$.

Define

$$\Delta a^{p^n} = \frac{a^{p^{n+1}} - a^{p^n}}{p^{n+1}}, \qquad \Delta^2 a^{p^n} = \frac{\Delta a^{p^{n+1}} - \Delta a^{p^n}}{p^{n+1}}$$

and generally

$$\Delta^{s+1} a^{p^n} = \frac{\Delta^s a^{p^{n+1}} - \Delta^s a^{p^n}}{p^{n+1}}.$$

In $\Delta_r^{(m)}$ subtract the jth column from the $(j+1)$th for $j = 0, 1, \cdots, r-1$.

Then

$$\Delta_r^{(m)} = p^{mr+r(r+1)/2} \left| a_i^{p^m} \Delta a_i^{p^m} \cdots \Delta a_i^{p^{m+r-1}} \right| \quad (i = 0, 1, \cdots, r).$$

Again subtracting the jth column from the $(j+1)$th for $j=1, 2, \cdots, r-1$, we get

$$\Delta_r^{(m)} = p^e \left| a_i^{p^m} \Delta a_i^{p^m} \Delta^2 a_i^{p^m} \cdots \Delta^2 a_i^{p^{m+r-2}} \right|,$$

where $c=m(2r-1)+r(r+1)/2+r(r-1)/2$. Continuing in this way we finally get

$$\Delta_r^{(m)} = p^e \left| \Delta^j a_i^{p^m} \right| \qquad (i, j = 0, 1; \cdots, r), \tag{2}$$

where

$$e = m\{r + (r-1) + \cdots + 1\} + \{r(r+1)/2 + (r-1)r/2 + \cdots + 1\}$$
$$= mr(r+1)/2 + r(r+1)(r+2)/6.$$

Now Schur (*Sitzungsberichte der Preussischen Akademie der Wissenschaften*, 1933, pp. 145–151) has proved that for all $(a, p)=1$, $r<p$, the numbers $\Delta^r a^n$ are integral for all $n \geqq 0$. Consequently (2) implies (1).

Also solved by F. W. Ponting.

Two Limits Suggested by Probabilistic Considerations

4646 [1955, 447]. *Proposed by Oliver Gross, the Rand Corporation*

Let $F_1(x)=G_1(x)=x$ for x real, and for $n>1$ define F_n, G_n recursively by

$$F_n(x_1, \cdots, x_n) = \max(x_1, G_{n-1}(x_2, \cdots, x_n)),$$
$$G_n(x_1, \cdots, x_n) = \min(x_1, F_{n-1}(x_2, \cdots, x_n)).$$

Are the following relations true:

$$\lim_{n\to\infty} \int_0^1 \cdots \int_0^1 F_n(x_1, \cdots, x_n) dx_1 \cdots dx_n = \frac{\pi\sqrt{3}}{9},$$

$$\lim_{n\to\infty} \int_0^1 \cdots \int_0^1 G_n(x_1, \cdots, x_n) dx_1 \cdots dx_n = 1 - \frac{\pi\sqrt{3}}{9}?$$

Solution (abridged) by Chih-yi Wang, University of Minnesota. The given limits are correct. Define

$$\int_0^1 \cdots \int_0^1 [F_n(x_1, \cdots, x_n)]^r dx_1 \cdots dx_n = I(r, n),$$

$$\int_0^1 \cdots \int_0^1 [G_n(x_1, \cdots, x_n)]^s dx_1 \cdots dx_n = J(s, n).$$

It can be easily verified that for $n>2$:

$$(1)\qquad I(r,n)=\frac{1}{r+1}+\frac{r}{r+1}I(r+1,n-2)-\frac{r}{r+2}I(r+2,n-2),$$

$$(2)\qquad J(r,n)=I(r,n-1)-\frac{r}{r+1}I(r+1,n-1),$$

$$(3)\qquad I(r,1)=\frac{1}{r+1},\qquad I(r,2)=\frac{1}{r+1}+\frac{r}{(r+1)(r+2)}.$$

Case I ($n=2m+1$). By applying (1) successively and by aid of (3) we obtain, omitting details,

$$I(1,n)=\int_0^1\int_0^y\frac{1-x^{m+1}(1-x)^{m+1}}{1-x+x^2}\,dxdy.$$

In this integral, the part which is independent of m yields the required result $\pi\sqrt{3}/9$, while the other part can be so estimated that the lower bound of the denominator is $3/4$ and the upper bound of the numerator is 4^{-m-1} which will $\to 0$ as $m\to\infty$.

Case II ($n=2m$). By applying (1) successively and by aid of (3) we obtain, again omitting details,

$$I(1,n)=\int_0^1\int_0^y\frac{1-x^m(1-x)^m}{1-x+x^2}\,dxdy+\int_0^1\int_0^y x^m(1-x)^{m-1}dxdy.$$

The first integral converges to the same limit $\pi\sqrt{3}/9$ as stated in case I, and the integrand of the second integral is dominated by 4^{-m+1} which will $\to 0$ as $m\to\infty$.

The second proposed statement can be verified by (2) and the first result.

The result of the problem can be generalized as follows: For any given positive integers r and s, we have

$$\lim_{n\to\infty}I(r,n)=\int_0^1\int_0^y\frac{rx^{r-1}}{1-x+x^2}\,dxdy,$$

$$\lim_{n\to\infty}J(s,n)=\int_0^1\int_0^y\frac{sx^{s-1}(1-x)}{1-x+x^2}\,dxdy,$$

which can be verified in a straightforward manner.

Note by the Proposer. The following heuristic argument is of interest. Interpret the x_i's as independent random variables from the uniform distribution over $[0,1]$. The integrals in question represent the expected values of the two random variables F_n, G_n. But since, if x_1 and x_2 are independent random variables, it is easy to see that

$$P\{\max(x_1,x_2)\leqq x\}=P\{x_1\leqq x\}\cdot P\{x_2\leqq x\},$$

$$P\{\min(x_1,x_2)\leqq x\}=1-\{1-P(x_1\leqq x)\}\{1-P(x_2\leqq x)\},$$

whence we can set up a pair of simultaneous equations for the limiting distributions f, g of F_n, G_n (which can be shown to exist), namely,

$$f(x) = x \cdot g(x), \qquad g(x) = 1 - (1 - x)(1 - f(x)), \qquad x \in [0, 1].$$

Then

$$f(x) = \frac{x^2}{1 - x + x^2} \quad \text{and} \quad g(x) = \frac{x}{1 - x + x^2}.$$

The limits required by the problem are then given by

$$\int_0^1 x df(x) \quad \text{and} \quad \int_0^1 x dg(x)$$

which give the numerical values as stated.

Also solved by Richard Courter, M. L. Freimer and A. L. Tritter, R. D. Gordon, and E. C. Milner.

RECENT PUBLICATIONS

All new material for this section should be sent to the incoming editor, Richard V. Andree, University of Oklahoma, Norman, Oklahoma.

Elementary Topology. By D. W. Hall and G. L. Spencer, II. New York, John Wiley & Sons, Inc., 1955. 7+303 pages. $7.00.

As point set topology has become firmly established in a position of fundamental importance in much of modern mathematics, it is inevitable that attention be given to the feasibility and desirability of introducing the subject to the student of mathematics at an early stage in his education. At many schools, it is difficult, if not impossible, to teach an elementary course, unless one has available an adequate text. Those who feel that topology should be taught early will probably welcome the book by Hall and Spencer, for this book is intended as a text for an introductory course in point set topology for students who have not necessarily had courses in advanced calculus and modern algebra. The book does not treat algebraic topology.

The material treated in the first half of the book (149 pages) is elementary and should prove to be accessible to undergraduates. The second half of the book, which is primarily concerned with the proofs of some of the gems of topology, is, in this reviewer's opinion, too difficult for the typical undergraduate mathematics major. Nevertheless, the authors hope that it will be accessible, and therefore spell out all proofs in great detail.

The book begins with a concise chapter on set theory, which, with minor exceptions, meets the prevailing standards of rigor for non-axiomatic set theory.

Chapter II is devoted to a treatment of the basic properties of the real number system. Concepts such as connectedness and continuity are defined and developed in such a way as to prepare the student for the corresponding abstractions he will meet in later chapters. Presumably in an attempt to keep the book elementary, no definition of the real numbers is given. It is assumed that each real number has a "representation" as an infinite decimal. The reviewer believes that some students will feel uncomfortable with this treatment.

Chapters III and IV, which comprise about 100 pages, are devoted to topological and metric spaces. The basic concepts are here introduced, almost always with their most modern definitions, and the fundamental theorems are carefully proven. There are a large number of exercises, some of which are routine, others of which are thought-provoking. Some topics not treated in the text, notably the theory of category and of Moore-Smith convergence, are developed in a series of exercises. The contents of Chapters III and IV will probably be the heart of any introductory course in point set topology.

The remainder of the book is concerned with some beautiful but deep and difficult concepts and theorems. The unit interval and circle are given topological characterizations along the lines first done by Janiszewski. The famous theorem of Hahn and Mazurkiewicz which characterizes the continuous image of the closed unit interval, as well as Zippin's characterization of the simple closed surface are presented. Also proven are the Jordan curve theorem, the Jordan-Schoenflies theorem, and theorems concerning complex concepts introduced by Sierpinski and Bing. The last chapter of the book deals with transfinite reasoning, *i.e.*, the Axiom of Choice, Zorn's lemma, well-ordering, and Tychonoff's theorem.

Some readers of the book may be critical of some aspects of the authors' style. For example, some may feel that the authors should have presented Hilbert's proof of Peano's startling discovery that the closed unit square is the continuous image of the unit interval in order to elucidate the statement and proof of the more general theorem of Hahn and Mazurkiewicz. Others, however, may feel grateful that the authors left to the instructor pleasant opportunities to delight and enlighten his class. In any event *Elementary Topology* is a welcome addition to the relatively few books suitable as texts for courses for advanced undergraduates in modern mathematics.

L. E. DUBINS
Carnegie Institute of Technology

Through the Mathescope. By C. S. Ogilvy. New York, Oxford University Press 1956. 162 pages. $4.00.

The current importance of mathematics is generally acknowledged: it is an indispensable adjunct to technology. But to the public this merely signifies that a familiar evil has become a necessary one. The greater therefore is the mathematicians' wish "to proclaim to the world that their beloved mathe-

matics is not the dry-as-dust, ugly, routine affair that so many people think it is."

The author, like others before him, accepts the challenge. But he is unlike them in at least some of these respects: he speaks briefly and on few but well-chosen topics; he does not assume that a reader with little mathematical preparation may safely be treated like a fool; he provides elementary proofs when available and none otherwise; he distinguishes nicely between mathematical truth and physical reality; he exploits some of the simpler paradoxical situations; he makes a single concept do multiple duty whenever possible—thus providing a happy continuity between otherwise apparently unrelated subjects; he injects lively historical observations; he stresses the artistic element; and by his breezy style no less than the remarkable recency of many of his references he makes mathematics live.

The "mathescope," says the author (after crediting invention of the word to Watson Davis), is the mathematician's counterpart of the scientists' telescope and microscope, "with reason for its pedestal and inspiration for its lenses." Subjects viewed through the mathescope are organized into 11 intriguingly titled chapters (*e.g.*, "Wheels within Wheels" for the one on cycloids) and include (though not in this order) integral, rational, irrational, imaginary, and transfinite numbers; Pythagoras's Theorem, the regular polyhedra, coordinate geometry, the conic sections, and cycloids; infinite series and continued fractions; fallacies arising from algebraic misdemeanors, pictorial proofs, and immoderate enthusiasm for circle-squaring and angle-trisecting; probability, with the binomial and normal distributions; and elementary topology.

An appendix of 23 pages provides a running commentary on the 129 pages of main text, with additional references, proofs of slightly higher order of difficulty than those presented in the earlier section, and what might be termed "lay research questions".

The only statement in the book to which the reviewer took serious exception was this: "Essentially what the calculus does is integrate (sum) an infinite number of infinitely small elements of this type" (p. 86). However, the author is still apologizing for it two pages later. (One recalls with a shudder the presentations of the calculus offered by authors like Hogben!) Misprints noted were "x^2" for "p^2" on p. 143, missing subscripts on a couple of "V's" on p. 144, and a superfluous "the" (within a quotation) on p. 118. Slips in reasoning, terminology, *etc.* which were observed were few in number and were generally slight.

Perhaps the strongest feature of the book is its success in conveying the author's personal enthusiasm for mathematics. It is hard to imagine anyone's reading a single page without concluding that mathematics is a subject in which at least Professor Ogilvy is passionately interested! The more laymen who can be tempted to read *Through the Mathescope* the better.

H. J. HAMILTON
Pomona College

Methods in Numerical Analysis. By Kaj L. Nielsen. New York, The Macmillan Company, 1956. xiii+382 pages. $6.90.

It has become increasingly evident that numerical analysis may now be regarded as having a *raison d'être* in the modern mathematics curriculum.

Dr. Nielsen has sought to write *Methods in Numerical Analysis* as an elementary textbook and has approached his objective in a careful and effective manner. As the title implies, the subject matter stresses methodology in the art of computation rather than mathematical sophistication. Appropriate emphasis is placed on those modern computational procedures arising in the analysis of tabulated data and approximate solutions of equations which lend themselves to the use of tables included in the appendix, or are easily adapted to automatic desk calculators for problem laboratory purposes.

The first five chapters are concerned primarily with classical material from the theory of errors, accuracy of approximate calculations, finite differences, interpolation theory, numerical differentiation and integration. Chapters six and seven deal principally with existent procedures for numerical solutions of sets of linear and nonlinear algebraic equations, and an introduction to techniques for handling differential, difference, and partial differential equations. Chapters eight and nine portray some recent developments in the representation of empirical functions and smoothing of data employing the methods of least squares, Nielsen-Goldstein, orthogonal polynomials, and autocorrelation functions.

The author apparently believes firmly in systematic procedures and utilizes a multitude of calculating forms and schematics in the topical illustrations which are worked out in detail.

In summary, it is the reviewer's opinion that the author has succeeded in writing an effective elementary text and convenient reference on up-to-date methods in the field of numerical analysis, useful for a wide variety of situations encountered by students and workers in applied mathematics.

W. E. RESTEMEYER
University of Cincinnati

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

ORACLE APPLICATIONS PROGRAM

The Oak Ridge Institute of Nuclear Studies has announced that it will make available to universities throughout the region the services of the ORACLE, its

high-speed electronic digital computer. Dr. T. W. Hildebrandt will direct the new program. One object of the program is to aid in extending to university campuses a wider awareness of the capabilities of modern high-speed computing equipment. The program will also assist university research programs.

For further information write to Dr. T. W. Hildebrandt, ORACLE Applications, University Relations Division, Oak Ridge Institute of Nuclear Studies, P.O. Box 117, Oak Ridge, Tennessee.

THE UNIVERSITY OF ILLINOIS PROGRAM OF HIGH SCHOOL MATHEMATICS

The University of Illinois has announced that a grant of $277,000 from the Carnegie Corporation of New York will be used by a University committee to continue its work of devising a new mathematics course for each of the four secondary years, preparing classroom text materials and teachers' manuals, and conducting training courses to acquaint teachers with the new teaching techniques to be developed. This committee consists of representatives of the colleges of education, engineering, and liberal arts and sciences. The Illinois project has the goal of bringing the high school mathematics curriculum up to date by including topics from modern mathematics and by presenting the subject as an integrated body rather than as a group of isolated courses.

PERSONAL ITEMS

Professor Richard Courant, head of New York University's Graduate Mathematics Department and scientific director of the Institute of Mathematical Sciences, has been elected to foreign membership in the Royal Netherlands Academy of Sciences and Letters.

Professor S. C. Kleene of the University of Wisconsin has been elected President of the Association for Symbolic Logic for a three-year term beginning January 1, 1956.

Assistant Professor M. D. Marcus of the University of British Columbia has been awarded a Postdoctoral Research Associateship for advanced study at the National Bureau of Standards.

Professor J. J. Stoker, Jr., associate director of the Institute of Mathematical Sciences, New York University, has been awarded the Dannie Heineman Prize of $5,000 for his two books, "Non-Linear Vibrations" and "Theory of Water Waves".

California Institute of Technology announces the following: Professor H. F. Bohnenblust has been appointed Dean of Graduate Studies; Associate Professor C. R. DePrima has been promoted to a professorship in the Department of Applied Mechanics; Assistant Professor T. M. Apostol has been promoted to an associate professorship.

At Purdue University: Professor C. Y. Pauc of the Faculté des Sciences de Rennes, France, has been appointed to a visiting professorship; Dr. H. S. C. Sharp, previously chairman of the Department of Mathematics of the U. S. Coast Guard Academy, has been appointed to an associate professorship at

the Hammond Extension Center; Dr. Melvin Carter, formerly a research assistant at North Carolina State College, Professor Philip Dwinger of the University of Indonesia, Assistant Professor Imanuel Marx of the University of Michigan, and Dr. R. F. Williams, previously a visiting lecturer at the University of Wisconsin, have been appointed to assistant professorships; Dr. M. J. Mansfield, previously a research assistant at the University, has been appointed to an instructorship.

Seattle University announces that Mr. C. C. Chang and Mr. A. L. Yandl have been appointed to instructorships.

University of California, Davis, reports: Assistant Professor Anne C. Davis of the University of Washington has been appointed to an assistant professorship; Professor G. A. Baker has been named Faculty Research Lecturer.

University of Washington (Seattle) announces: Assistant Professors M. G. Arsove, F. H. Brownell, and E. A. Michael have been promoted to associate professorships; Instructors H. A. Forrester, J. H. Walter, and R. L. Vaught have been promoted to assistant professorships; Associate Professor V. L. Klee, Jr. has been awarded a Sloan Foundation Fellowship; Assistant Professor R. L. Vaught has been awarded a Fulbright Fellowship for study during 1956–57 at the University of Amsterdam; Associate Professor E. A. Michael will spend the year 1956–57 at the Institute for Advanced Study; J. Butler of the University of California at Los Angeles, R. K. Getoor of Princeton University, T. J. McMinn of the University of California at Berkeley, and A. Nijenhuis of the University of Chicago have been appointed to assistant professorships; R. M. Blumenthal of Cornell University, G. Helmberg of Innsbruck, Austria, and R. T. Ives of the University of Washington have been appointed to instructorships; J. M. G. Fell of the University of Chicago has been appointed to a research assistant professorship under a grant from the Office of Naval Research.

Wayne State University announces: Professor S. T. Hu of the University of Georgia and Professor Wallace Givens of the University of Tennessee have been appointed to professorships; Assistant Professor Felix Haas, University of Connecticut, and Assistant Professor M. T. Wechsler of the State College of Washington have been appointed assistant professors.

Dr. S. S. Abhyankar, recently an associate at Columbia University, has been appointed to a visiting assistant professorship at Columbia University.

Assistant Professor A. V. Banes of Trinity University has a position as a programmer analyst with the Ramo-Wooldridge Corporation, Los Angeles, California.

Mr. C. R. Berndtson, previously a teacher at Vinalhaven High School, Maine, is employed as an applied research mathematician by the Avco Manufacturing Corporation, Stratford, Connecticut.

Dr. F. G. Brauer, formerly a graduate student at Massachusetts Institute of Technology, has been appointed to an instructorship at the University of Chicago.

Professor T. H. Brown, Graduate School of Business Administration, Har-

vard University, has been appointed to an emeritus professorship in business statistics.

Dr. R. G. Buschman of the University of Colorado has been appointed to an associate professorship at the University of Wichita.

Assistant Professor Lorenzo Calabi of Boston College has been promoted to an associate professorship.

Mr. M. N. Chase, previously a mathematician at Battelle Memorial Institute, has accepted a research position with Technical Operations, Inc., Arlington, Massachusetts.

Mr. T. E. Cheatham, Jr., formerly a research assistant at Purdue University, is Head of the Computer Division, Technical Operations, Inc., Washington, D. C.

Dr. John Christopher of Fresno State College has a position as a mathematician with the ElectroData Corporation, Pasadena, California.

Assistant Professor L. J. Cote of Purdue University has been appointed to an assistant professorship at Syracuse University.

Mr. D. F. Coulter, Jr. of Grays Harbor College has been appointed to an instructorship at Hartnell College.

Mr. D. L. Daly of Missouri School of Mines and Metallurgy has been appointed to an assistant professorship at Illinois Wesleyan University.

Mr. D. C. Davis, previously a development engineer for McDonnell Aircraft Corporation, St. Louis, Missouri, is employed now as an engineer by Chance-Vought Aircraft, Dallas, Texas.

Dr. Bernard Dimsdale of Raytheon Manufacturing Company has accepted a position as Director of the Univac Computing Center, Remington Rand Division of Sperry-Rand Corporation, Los Angeles, California.

Dr. Jesse Douglas of Columbia University has been appointed to a professorship at the City College of the City of New York.

Dr. H. T. Engstrom, formerly vice-president for Research and Engineering, Engineering Research Associates, Washington, D. C., is Vice-President of Remington-Rand, Division of Sperry Rand Corporation, New York, New York.

Assistant Professor M. P. Epstein of Johns Hopkins University has received an appointment as mathematician at Bell Telephone Laboratories, Whippany, New Jersey.

Mr. D. O. Etter, Jr., of International Lubricant Corporation, New Orleans, Louisiana, has accepted a position as a research engineer with North American Aviation, Los Angeles, California.

Visiting Assistant Professor G. H. F. Gardner of Carnegie Institute of Technology has a position as a mathematical physicist with Gulf Research and Development, Pittsburgh, Pennsylvania.

Mr. L. T. Gardner, formerly an actuarial analyst for the Travelers Insurance Company, Hartford, Connecticut, is teaching at Oakwood School, Poughkeepsie, New York.

Professor A. W. Goodman of the University of Kentucky is on leave at the Institute for Advanced Study.

Dr. Edison Greer, previously a design engineer at Aerojet General, Azusa, California, and professor of mathematics at Mount San Antonio College, has a position as a research scientist on the applied mathematics staff of Missile Systems Division Research Laboratories, Lockheed Aircraft Corporation, Palo Alto, California.

Professor I. R. Hershner, Jr., formerly chairman of the Department of Mathematics of the University of Vermont, has accepted a position with the Office of Chief of Research and Development, U. S. Army, Washington, D. C.

Professor Ralph Hull, head of the Department of Mathematics of Purdue University, has accepted a position as a staff member of the Ramo-Wooldridge Corporation, Los Angeles, California.

Associate Professor P. E. Irick of Purdue University has accepted a position with the Highway Research Board's Road Test Project.

Assistant Professor H. G. Jacob, Jr., of Louisiana State University has been promoted to an associate professorship.

Mr. Edgar Karst, formerly an electronic computing analyst with Great Lakes Pipe Line Company, Kansas City, Missouri, now has a position as a computing engineer with the IBM Scientific Computation Laboratory, Endicott, New York.

Assistant Professor N. D. Kazarinoff of Purdue University has been appointed to an assistant professorship at the University of Michigan.

Mr. J. P. Line of the University of Rochester has been appointed to an assistant professorship at Georgia Institute of Technology.

Mr. G. E. Mahoney of Computer Control Company, Inc., Point Mugu, California, has accepted a position as Director of the Computing Bureau, SPICA, Inc., Somerville, Massachusetts.

Dr. P. J. McCarthy of the College of the Holy Cross has been appointed to an assistant professorship at Florida State University.

Mr. D. S. McManus of the University of Buffalo has been appointed Assistant Professor of Mechanical Engineering at Norwich University.

Mr. D. D. Morrison, previously a research associate in the Computation Laboratory, Wayne State University, has a position as a member of the technical staff of the Ramo-Wooldridge Corporation, Los Angeles, California.

Dr. Margaret Owchar, formerly a researcher and statistician for the Manitoba Cancer Relief and Research Institute, Winnipeg, Manitoba, has been appointed to an assistant professorship at Bemidji State Teachers College.

Associate Professor P. M. Pepper of Ohio State University is now Director of the Mapping and Charting Research Laboratory of the University.

Assistant Professor A. J. Perlis of Purdue University has been appointed to an associate professorship at Carnegie Institute of Technology.

Mr. Arthur Reetz, Jr., of Missouri School of Mines has a position as a nu-

clear engineer with Consolidated-Vultee Aircraft Corporation, Fort Worth, Texas.

Assistant Professor J. B. Roberts of Reed College is on leave of absence for the academic year 1956–57 as Visiting Assistant Professor at Wesleyan University.

Dr. Berthold Schweizer has been appointed to an instructorship at Illinois Institute of Technology.

Mr. Seymour Sherman of Research and Advanced Development Division, Avco Manufacturing Company, Stratford, Connecticut, has accepted a position as staff member of Dian Laboratories, New York, New York.

Professor C. N. Shuster, head of the Department of Mathematics of State Teachers College, Trenton, New Jersey, has retired from this position. He is teaching at Trenton Junior College and at The Pennington School, Pennington, New Jersey.

Mr. W. K. Spears of the Electronics Corporation of America has accepted a position as an engineer with the General Electric Company, Utica, New York.

Assistant Professor E. A. Sturley of Allegheny College has been promoted to an associate professorship.

Professor Irving Sussman of California State Polytechnic College has been appointed Professor and Chairman of the Department of Mathematics at the University of Santa Clara.

Professor H. L. Turrittin of the University of Minnesota was awarded a Fulbright Fellowship and was Visiting Professor at the University of Innsbruck, Austria, during 1955–1956.

Associate Professor W. R. Utz, Jr., of the University of Missouri is on leave of absence for the year 1956–57 and is spending the year at the Institute for Advanced Study.

Dr. Hermann von Schelling, formerly a biomathematician for the Naval Medical Research Laboratory, U. S. Submarine Base, New London, Connecticut, has a position as a consultant mathematician with the General Electric Company, Schenectady, New York.

Miss Ruth A. Wagner of the Rand Corporation has accepted a position as a mathematician with the Ramo-Wooldridge Corporation, Los Angeles, California.

Assistant Professor L. E. Ward, Jr., of the University of Utah has accepted a position as a mathematician with the U. S. Naval Ordnance Test Station, China Lake, California.

Dr. Jack Warga of Consolidated Engineering Corporation, Pasadena, California, is employed as a mathematician by the ElectroData Corporation, Pasadena.

Mr. C. S. Wolfe of the University of Maryland has been appointed to an assistant professorship at the U. S. Naval Academy.

Mr. F. W. Wolock of Iona College has been appointed to an instructorship at St. John's University.

Professor E. J. Lowry, head of the Department of Mathematics of Hastings College, died on February 18, 1956.

Professor J. A. McKee of Beaver College died on February 7, 1956.

Dean Emeritus T. G. Rodgers of New Mexico Highlands University died on May 19, 1956. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 84 persons have been elected to membership by the Board of Governors on applications duly certified.

J. G. ALEX, A.B. (Sacramento S.C.) Math., University of California Radiation Lab.

L. J. ANDREWS, B.S. (U.C.L.A.) Project Engr., National Cash Register Co., Hawthorne, Calif.

G. C. ARROWSMITH, Student, Oregon State College.

A. C. BARBEAU, Student, University of Montreal.

R. J. BEIL, M.S. (Okla. A.&M.C.) Asst. Professor, Vanderbilt University.

JOAN BLEWITT, Student, Seton Hill College.

C. O. BLOOM, M.S. (Pennsylvania) Math. Teacher, Central School, Ellenville, N. Y.

D. M. BLOOM, Student, Columbia University.

M. L. BROUSSARD, Student, Southwestern Louisiana Institute.

PATRICIA A. BROWN, Student, Trinity University.

D. J. C. BURES, Student, Queen's University.

P. B. BURT, Student, University of Tennessee.

E. W. CAMPBELL, A.B. (San Jose S.C.) Math., University of California Radiation Lab.

F. J. CARTER, Student, LeMoyne College.

C. C. CONLEY, Student, Wayne University.

W. C. CREHL, M.A. in Ed. (Arizona S.C., Flagstaff) Math., Holloman Air Force Base, N. Mex.

C. F. DANIEL, Student, Oregon State College.

J. K. DIEI, Student, Ottawa University.

THADDEUS DILLON, M.S. (John Carroll) Instr., John Carroll University.

IRMA B. ESRIG, Student, City College of the City of New York.

J. C. EZYK, A.B. (St. Anselm's) Asst. Professor, St. Anselm's College.

J. B. FLANSBURG, B.S. (New Hampshire) Chairman, Department of Mathematics, St. John's School, Houston, Texas.

F. E. GINSBERG, Instr., Brooklyn Polytechnic Institute.

ALVA M. GIRDLEY, Student, Cardinal Stritch College.

W. G. GOELLER, Central Office Tech., New York Telephone Co., N. Y.

W. E. GOULD, Student, Rutgers University.

E. F. GRIFFIN, II, Student, Georgia Institute of Technology.

R. E. GRISWOLD, Student, Stanford University.

D. E. HARRIS, JR., B.S. (Southern Methodist) Staff Member, University of California, Los Alamos Scientific Lab., N. Mex.

J. B. HARVILL, JR., B.A. (North Texas S.C.) Teaching Fellow, North Texas State College.

F. E. HAZLETT, B.A. (U.C.L.A.) Computer Applications Engr., Librascope, Glendale, Calif.

H. G. HENRY, Student, University of Oregon.

W. H. HOLTER, B.S. (Franklin and Marshall) Applied Math., Atlantic Research Corp., Alexandria, Va.

E. L. HUBBARD, Employee, Wright Studio and Camera Shop, Midland, Mich.

R. W. HUNT, Student, West Texas State College.

MEYER JORDAN, Ph.D. (N.Y.U.) Lecturer, Brooklyn College.

D. S. KAHN, Student, Princeton University.

J. L. KATZ, M.S. (Union C.) Engr., General Electric Co., Schenectady, N. Y.

DOROTHEA A. KELLER, Student, University of Rochester.

J. W. KENNEDY, B.S. (Purdue) Wisconsin High School, University of Wisconsin.

W. J. LEE, M.A. (Carleton) Asst. Teacher, Hunter College High School, New York, N. Y.

GERALD LEIBOWITZ, Student, City College of the City of New York.

I. J. LIEBERMAN, M.A. (Southern California) Sr. Res. Scientist, Litton Industries, Beverly Hills, Calif.

MARSHALL LUBAN, Student, Yeshiva University.

H. B. LYNCH, Student, Colorado School of Mines.

KURT MAHLER, D.Sc. (Manchester) Professor, Manchester University, England.

R. A. MAIER, B.A. (U.C.L.A.) Sr. Lofting Math., North American Aviation, Santa Monica, Calif.

E. V. MARTIN, Student, University of Cincinnati.

J. G. MAXWELL, M.A. (Kent S.U.) Teacher, John Marshall High School, Cleveland, Ohio.

REV. L. W. MCCALL, S.A.C., M.S. (Marquette) Supervisor, Pius XI High School, Milwaukee, Wisc.

M. C. MCCORD, Student, University of Tennessee.

M. A. MCKIERNAN, Ph.D. (Illinois I.T.) Instr., Illinois Institute of Technology.

MADELINE D. MESSNER, M.A. (Montclair S.T.C.) Teacher, Abraham Clark High School, Roselle, N.J.

R. L. MILLS, M.S. (Southern Methodist) Sr. Computing Engr., North American Aviation, Los Angeles, Calif.

CAROLEN MONCURE, M.A. (Fisk) Crystal Springs, Mississippi.

R. A. NEUMANN, Student, Knox College.

R. L. NORRIS, Student, Eastern Kentucky State College

BETTY OPPENHEIM, B.A. (Brooklyn C.) Math., Brookhaven National Lab., Upton, N. Y.

MR. JINGYAL PAK, Student, Kent State University.

H. W. PALCZEWSKI, A.M. (Columbia) Instr., City College of the City of New York.

C. W. PECK, B.S. (Trinity U.) Stock Clerk, Southwest Research Institute, San Antonio, Texas.

P. L. PHIPPS, M.S. (M.I.T.) Math., Remington Rand Univac, St. Paul, Minn.

E. A. PORTER, B.A. (Michigan) Vice President and Chief Actuary, Manhattan Life Insurance Co., New York, N. Y.

PHILBURN RATOOSH, Ph.D. (Columbia) Asso. Professor of Psychology, Ohio State University.

ALFRED REICHENTHAL, M.A. (Columbia) Res. Engr., Hughes Aircraft Co., Los Angeles, Calif.

MICHAEL ROSEN, Student, Brandeis University.

L. P. ROSENBERRY, B.S. (Pennsylvania S.U.) Applied Science Representative, International Business Machines, Detroit, Mich.

D. G. RUSSELL, M.A. (Oklahoma) Exploitation Engr., Shell Oil Co., Odessa, Texas.

R. E. SCHLEA, M.A. (Bowling Green S.U.) Instr., Ohio Northern University.

HERMANN SCHMIDT, Ph.D. (Jena) Professor, Universität Wurzbürg, Germany.

JANET H. SCOTT, B.S. (Limestone) Grad. Asst., University of South Carolina.

J. J. SEGEDY, Student, Queens College.

C. P. SEGUIN, Student, University of Detroit.

INEZ SHAPIRO, Student, Northeastern University.

E. A. SHULMAN, Student, McGill University.

A. J. STRACCIOLINI, B.A. (Temple U.) Math., Ballistic Research Lab., Aberdeen, Md.

E. H. SWIFT, M.A. (Southern California) Chairman, Department of Mathematics, El Cerrito High School, Calif.

MYLES TIERNEY, Student, Brown University.

T. L. TRUEMAN, Student, Dartmouth College.

R. C. WARTH, Student, University of Rochester.

MR. CHIEN WENJEN, Ph.D. (U.C.L.A.) Asst. Professor, Texas Technological College.

T. L. WILLIAMS, M.S. (Utah) Instr., Idaho State College.

R. J. WINKELMAN, Student, University of Rochester.

B. A. YALE, Student, University of Buffalo.

THE APRIL MEETING OF THE KANSAS SECTION

The forty-first annual meeting of the Kansas Section of the Mathematical Association of America was held at the University of Wichita, Wichita, Kansas, on April 21, 1956, in conjunction with the annual meeting of the Kansas Association of Teachers of Mathematics. There were 130 persons registered, including 40 members of the Association. Sessions of the Association were held on Saturday morning and afternoon. Professor Ferna Wrestler, Chairman, presided.

The following officers were elected for one-year terms: Chairman, Professor W. R. Scott, University of Kansas; Vice-Chairman, Professor L. E. Laird, Kansas State Teachers College, Emporia; Secretary-Treasurer, Miss Helen F. Kriegsman, Kansas State Teachers College, Pittsburg.

By invitation of the Executive Committee, Professor B. W. Jones of the University of Colorado delivered an address at the morning session entitled "Observations on English Schools and Colleges."

The following short papers were presented at the afternoon session:

1. *Quality control*, by Professor F. C. German, Kansas State Teachers College, Pittsburg, introduced by the Chairman.

As tolerances narrow and the demands of the general public become more exacting, the demand for a higher and more uniform quality and for more refined engineering procedures grows at an astounding rate. Statistical quality control is the tool used to maintain and improve this quality. The established distributions such as the normal, Poisson's approximation, or the log normal, have wide enough practical applicability to approximate a vast number of situations that arise. In other instances a distribution must be established empirically in which case its validity is restricted to the particular type of procedure being studied. It is an obligation imposed on the control operator that he keep in constant touch with the frontier of mathematical development as it pertains to his field.

2. *Divided differences and linear differential equations*, by Dr. A. H. Kruse, University of Kansas.

Let $\xi_x(m) = e^{mx}$ for all real x and complex m, let $m_0, m_1 \cdots m_n$ be complex numbers, and let $[m_0, \cdots, m_n; \xi_x]$ be the nth order divided difference of ξ_x at $(m_0, m_1, \cdots, m_n)$. If g is a real-valued continuous function on $a \leqq x \leqq b$, then $y(x) = \int_0^{x-a} [m_0, m_1, \cdots, m_n; \xi_u] g(x-u)du$ is that solution of the differential equation $(d/dx - m_0)(d/dx - m_1) \cdots (d/dx - m_n)y(x) = g(x)$ on $a \leqq x \leqq b$ such that $y^{(j)}(a) = 0$ for $0 \leqq j \leqq n$. The divided difference in the integrand leads to the inequality $|y(x)| \leqq 1/n! \int_0^{x-a} u^n e^{ru} |g(x-u)|du$, where r is the maximum of the real parts of $m_0, m_1, \cdots, m_n$.

3. *Geodesics in metric spaces*, by Professor W. L. Stamey, Kansas State College.

A geodesic in a metric space is a locally congruent map of the real axis. In a finitely compact, convex metric space in which each metric segment admits a unique local prolongation beyond each end point, each two points are joined by a geodesic. The nature of the geodesics in some such spaces is discussed, and some general properties of geodesics are pointed out.

4. *A note on teaching statement problems*, by Professor Paul Eberhart, Washburn University.

On a test in intermediate algebra it was found that most of the students could not, or did not, complete correctly a statement of the form "Let $x =$ ———," as the first step in a statement

problem. In spite of this, correct numerical answers were generally obtained. It is then fairly obvious why more difficult problems cause trouble. However, with some extra stress on the formulation of problems much better results were obtained.

5. *Report of Board of Governors Meetings*, by Professor C. B. Read, University of Wichita.

Professor Read reported on meetings of the Board of Governors held in Ann Arbor on August 29, 1955, and in Houston on December 29, 1955. Topics mentioned included the continuation of the program of visiting lecturers; the employment register; policies relative to solicitation of funds; Carus Monographs; omitting publication of names of members attending meetings; revised policies relative to payment of expenses of sections and of section officers attending annual meetings; high school contests; time of annual meetings.

LAURA Z. GREENE, *Secretary*

THE APRIL MEETING OF THE OHIO SECTION

The fortieth annual meeting of the Ohio Section of the Mathematical Association of America was held at Oberlin College, Oberlin, Ohio, on April 14, 1956. Professor R. R. Stoll, Chairman of the Section, presided at the morning and afternoon sessions.

One hundred twenty persons registered in attendance including eight-five members of the Association.

The following officers were elected for the coming year: Chairman, Professor P. V. Reichelderfer, Ohio State University; Secretary-Treasurer, Professor Foster Brooks, Kent State University; Third member of the Executive Committee, Professor W. E. Deskins, Ohio State University; Program Committee: Chairman, Professor H. D. Lipsich, University of Cincinnati, Professor E. B. Leach, Case Institute of Technology, Professor Sam Selby, University of Akron.

Professor C. H. Yeaton submitted a tribute to the late Professor W. D. Cairns, which was unanimously adopted by the Section.

The following papers were presented:

1. *Sperner's lemma*, by Professor R. R. Stoll, Oberlin College (Chairman's address).

Several fundamental theorems of topology can be proved in an entirely elementary way using the following combinatorial lemma due to Sperner: If the simplex S with vertices $P_0, P_1, \cdots, P_n$ is divided simplicially and each vertex e of every subsimplex T is assigned a number $n(e)$ such that if e lies on the face of S with vertices $P_{i_0}, P_{i_1}, \cdots, P_{i_s}$ then $n(e)$ is one of $i_0, i_1, \cdots, i_s$, then there exists an odd number of T's having vertices labeled with $0, 1, \cdots, n$. This result was proved and then applied to prove Lebesgue's Covering Theorem and Brouwer's Fixed Point Theorem.

2. *A proof of the composite function theorem for matric functions*, by Mr. D. W. Robinson, Case Institute of Technology.

The composite function theorem for matric functions was stated and demonstrated by R. F. Rinehart in this MONTHLY, vol. 62, 1955. The proof contained in that paper is rather lengthy and tedious. The present note provides a simpler proof, which depends upon the combinatorial properties of matric functions.

3. *Some applications of stochastic processes*, by Professor W. R. Van Voorhis, Fenn College.

Most of the developments in the theory of stochastic processes have taken place during the past 25 years. In certain applications, dynamic probability models of physical processes have been developed and described mathematically. Professor Van Voorhis pointed out that these models, used extensively in the field of Operations Research, arise out of sets of postulates concerning probabilities associated with a stochastic variable and characterizing the process. Examples were given of the use of the theory in problems involving contagion, "birth and death", queueing, and replacement.

4. *The theory of verniers on non-uniform graduated scales*, by Professor P. M. Pepper, The Ohio State University.

Since the time of Pierre Vernier (1580–1637) the application of a small auxiliary scale along a uniformly graduated scale to obtain refined reading has been in widespread use. Because of the non-uniformity it has been thought impossible to obtain a vernier action for any scales excepting those with equal graduations. The author shows that a vernier mechanism can be constructed for any scale generated by a strictly monotone continuous function. Such a vernier mechanism will be operable with all of the accuracy of an ordinary vernier. The nearest prior approach to verniers of non-uniform scales was disclosed in a British patent issued in 1944, to an inventor who laid out a scale for a logarithmic function on a suitably chosen logarithmic spiral for which the scale graduations have equal intervals of arc between them. Suitably spaced vernier lines were then drawn on a transparent pivoted arm in such a way that these lines cut off equal arc length intervals of the original scale line suitable for forming a vernier at each stage. The present author shows that this restriction, namely, of adapting the scale line to the function scale, is inessential to the construction of a correct vernier. A number of possible applications of the vernier device to instruments are cited, and the possibility of constructing a five significant figure slide rule with verniers for all numerical and trigonometrical scales is demonstrated. The author holds U. S. Patent No. 2,564,227 which covers, among other things, the application of verniers to non-uniform scales.

5. *The role of the mathematician in a computing laboratory*, by Mr. E. F. Ormsby, Applied Science Division, International Business Machines Corporation.

The rapid growth of computing laboratories in the past few years has opened wide opportunity for employment of mathematicians. A typical laboratory will employ from 25 to 200 mathematicians or physicists. All levels of professional qualification are required, from the new college graduate with a B.A. to the post graduate mathematician. Today's universities must expand their horizons to meet this challenge.

6. *The geometric method in modern algebra*, by Professor Ernst Snapper, Miami University. (By invitation.)

The geometric method in modern algebra consists of the geometric use of the notion of n-dimensional vector space with an arbitrary field as scalar domain. A good example of this method is Witt's theory of quadratic forms. A quadratic form is a metric space. The space is an n-dimensional vector space R with a commutative field F as scalar domain, where F does not have characteristic 2. The metric of R is given by an inner product for the vectors of R. Witt's theorem concerning the extension of isometries between subspaces of a nondegenerate R to isometries of R was discussed. Many other examples of the geometric method can, of course, be given. The talk ended with a discussion of how the fact, that the notion of n-dimensional vector space has become so important in algebra, should influence our undergraduate teaching.

7. *Some current problems of teachers of mathematics in elementary and secondary schools*, by Dr. Herschel Grime, Supervisor of Mathematics, Cleveland Public Schools. (By invitation.)

Due to the critical shortage of elementary teachers and mathematics teachers in the secondary schools, it is extremely difficult to maintain the integrity of mathematics courses. By assigning our best teachers to college preparatory mathematics classes this danger can be lessened to some extent. Careful guidance by the secondary schools insures that pupils having ability in mathematics elect courses best fitted to their ability and probable future needs. The courses in college preparatory mathemtics in the secondary schools are very much in need of reform. The necessary revisions and additions should be made by committees including both college and secondary teachers.

8. *Three courses in the elements of modern mathematics*, by Professors Gaylord Merriman and H. D. Lipsich, University of Cincinnati.

The authors discussed three courses to be given at the University of Cincinnati, designed to meet the problems raised by (a) the desire of local high schools to modernize their curricula, (b) the increasing number of graduates of honors programs in mathematics from the local high schools, (c) the insistence of social scientists on a revision of traditional freshman mathematics curriculum. These courses are respectively: (a) Fundamental Modern Concepts in Mathematics, For Teachers. —Some of the modern concepts now fundamental in the structure of secondary school and college mathematics prior to calculus, such as, Boolean algebra, basic algebraic structures, finite and non-euclidean geometries; matrix solution of systems of linear equations; (b) Tutorial Readings in Modern Mathematics.—Continuation of analytic geometry and calculus, introduction to the algebra of sets, logic and postulational systems, groups and matrices; (c) Fundamentals of Modern Mathematics.—Basic modern concepts now important in mathematics, science, and some social sciences, such as, logic and postulational systems, sets, groups, matrices; selected topics in algebra, trigonometry, analytic geometry, statistics and the ideas of calculus.

FOSTER BROOKS, *Secretary*

THE APRIL MEETING OF THE SOUTHWESTERN SECTION

The annual meeting of the Southwestern Section of the Mathematical Association of America was held at the New Mexico College of Agriculture and Mechanic Arts, State College, New Mexico, on April 27 and 28, 1956. Professor R. L. Westhafer, Chairman of the Section, presided at the afternoon session on April 27 and also at the morning session on April 28.

There were 76 persons in attendance including 31 members.

The following officers were elected: Chairman, Professor R. B. Lyon, Arizona State College at Tempe; Vice-Chairman, Dr. R. C. Hildner, Sandia Corporation, Sandia Base, Albuquerque, New Mexico; Secretary-Treasurer, Mr. W. W. Mitchell, Jr., Phoenix College.

The following papers were presented.

1. *The role of mathematics in present day technology*, by Dr. S. M. Ulam, Los Alamos Scientific Laboratory. (By invitation).

2. *A study on generalized implication*, by Professor D. L. Webb, University of Arizona.

The operations of implication, logical sum, logical product, and negation were generalized to cover n-values. A study was then made to see which of sixteen methods of proof listed by J. Barkley Rosser in his *Logic for Mathematicians* held in the generalized systems.

3. *First removal time in a general stochastic epidemic*, by Professor P. W. M. John, University of New Mexico.

In Bailey's model for a general stochastic epidemic a population of size n is considered, of which r are susceptibles, s are infectives (in circulation), and the remaining $z=n-r-s$ are either removed or immune. The probability that in an element of time dt a susceptible will become infected is $\beta rs dt+o(dt)$, and that an infective will be removed is $\gamma s dt+o(dt)$. The number of infectives in circulation is thus a birth and death process with $\lambda_s=\beta rs$ and $\mu_s=\gamma_s$. An expression is obtained for the expected time elapsed until the first removal after the population has reached the above state; the result is illustrated in the case of a small intra-household epidemic.

4. *Two basic methods in extended analytic geometry*, by Professor R. S. Underwood, Texas Technological College.

This paper describes briefly a flexible system for plotting on a plane the locus of an equation in n variables. When X and Y are specified single-valued functions of the variables, each set of coordinates determines a point (X, Y). Two types of plotting rules are found effective in solving simultaneous equations. (1) Simple rules may be used with which are associated many readily-found loci. (2) In a specific problem an artificial rule designed to make one locus a curve is often feasible, since the remaining loci may then be found by a routine method.

5. *A metric space defined on the subsets of* (0, 1), by Professor B. C. Meyer, University of Arizona.

Nikodym defined a metric space on the (Lebesgue) measurable subsets of (0, 1). This paper was concerned with extending his space to include nonmeasurable sets and examining the properties of the resulting space.

6. *The potential with Legendre polynomials for bodies with axis of symmetry*, by Professor O. B. Ader, New Mexico College of Agriculture and Mechanic Arts.

If a material body has an axis of symmetry, it is generally possible to express the potential $U(P)$, at any point P on this axis, as a function of r, the distance of the point from a fixed origin. When $U(P)$ is expanded in powers of r, a theorem from potential theory gives the potential at any point of free space as a series of Legendre polynomials. The theorem is illustrated for the case of a straight line segment of constant linear density, and the series compared with the closed form of the solution.

7. *Points at which the osculating conic hyperosculates a plane curve*, by Professor Louis Child, New Mexico College of Agriculture and Mechanic Arts.

At a point P of a plane curve Γ, the osculating conic γ_4 exists and then exists uniquely if and only if at P

(a) The radius of curvature ρ and its first two derivatives ρ_1, ρ_2 with respect to the arc-length s exist and are finite, and $\rho \neq 0$.

The conic γ_4 contacts Γ at P in at least six points if and only if (a) is true and also

(b) $$q = 36\rho_1 + 9\rho^3\rho_2 - 9\rho\rho_1\rho_2 + 4\rho_1^3 = 0.$$

The conic γ_4 contacts Γ at P in at least $n+6$ points if and only if (a) and (b) hold and also

(c) $$\frac{d^j q}{ds^j} = 0 \qquad 1 \leq j \leq n.$$

The derivatives (c) are the formal derivatives of (b), and (a), (b), (c) are all evaluated at P.

8. *More about the altitude quadric*, by Professor J. H. Butchart, Arizona State College at Flagstaff.

Professor Butchart showed that the quadric on which are the four altitudes of a tetrahedron is characterized by the properties that a section normal to any ruling is an equilateral hyperbola and that its asymptotic cone contains three mutually perpendicular generators. Analytically, a necessary and sufficient condition for a hyperboloid of one sheet to be an altitude quadric is that the sum of the coefficients of the square terms is zero. If the Monge point is equidistant from any three of the altitudes and orthocentric lines, it is equidistant from all, and these distances all lie in the equatorial plane of the quadric.

9. *Some mathematical problems investigated through computations on electric machines*, by Dr. S. M. Ulam, Los Alamos Scientific Laboratory. (By invitation).

As example of the heuristic value of calculations on fast computing machines, two problems were mentioned:

1. A study of the vibrating string with non-linear forces. The distribution of the energy among the various modes for the corresponding linear problem was studied as a function of time. A strange almost periodic behavior of the system was described. (A Los Alamos report by Fermi, Pasta, and Ulam in 1955 and a forthcoming one by Pasta and Ulam).

2. A study of a mathematical game by statistical experiments. (A paper by P. Stein and S. Ulam in the proceedings of the Louisiana Conference on Computors in 1955, and a forthcoming article by P. Walden in Volume 3 of Contributions to Theory of Games.)

10. *Making electrons count*, by Professor R. L. Westhafer, New Mexico College of Agriculture and Mechanic Arts.

Professor Westhafer showed a film on the Whirlwind Computer by courtesy of the Digital Computation Laboratory of the Massachusetts Institute of Technology.

11. *The down hill method of solving a polynomial $f(x)=0$, for real and imaginary roots*, by Dr. J. A. Ward, Holloman Air Force Base. (By invitation).

The author shows a numerical method of determining to any desired accuracy the real and imaginary roots of a polynomial equation with real coefficients. The technique of using this method on a small computer is also given.

12. *Some equations over complete Boolean algebras*, by Professor H. D. Sprinkle, University of Arizona, introduced by the Secretary.

The purpose of this paper is to establish as a main result necessary and sufficient conditions for β (β any ordinal number) equations in β unknowns to have a solution, not necessarily unique. These conditions will be stated in terms of the coefficients of the unknowns.

Other results—the existence of a minimal solution, a disjoint solution, and a unique solution—are stated for $\beta=2$, and the question as to generalizations is left open.

13. *Normal subgroups of direct products*, by Professor Ralph Crouch, New Mexico College of Agriculture and Mechanic Arts.

Let $\Sigma_m(H)$ be the monomial group of H over U where H is a group, U is a finite set. The normal subgroups of $\Sigma_m(H)$ are known (see: O. Ore, *Theory of Monomial Groups*, Trans. Amer. Math. Soc., vol. 51, 1942, pp. 15–64). The set of all monomial substitutions with permutation component in the alternating group is a subgroup $\Sigma_{A,m}(H)$ and the normal subgroups have been determined (see: R. B. Crouch, *Monomial Groups*, Trans. Amer. Math. Soc., vol. 80, pp. 187–215). If $m=2$, $\Sigma_{A,m}(H)$ is simply the direct product $H\times H$ and the structure theorems for the normal subgroups become a special case for the known theorems of Mazzoni (see: *Annali Della R. Scuola*

Normale Di Pisa, vol. 14) and Remak (see: *Journal fur die Reine und Angewändte Mathematik*, vol. 163, 1930.

14. *Intermediate algebra and an experiment*, by Professor F. C. Gentry, University of New Mexico.

Dr. Gentry pointed out that in view of the ever increasing number of students entering college who are unable to pass traditional courses in college algebra without remedial courses, it seems wise to find ways and means of removing these deficiencies as early as possible. He discussed an experiment with a combined course in intermediate and college algebra followed by a combined course in trigonometry and analytic geometry.

15. *An application of least squares*, by Mr. W. W. Mitchell, Jr., Phoenix College.

The method of least squares is used to determine a set f_i of n numbers and two other numbers, A and B, such that a set of n numbers, a_i, is approximated by Af_i, a set of n numbers, b_i, is approximated by Bf_i, and a third set of n numbers, c_i, is approximated by the set f_i, the approximations being the "best possible."

W. W. MITCHELL, JR., *Secretary*

THE APRIL MEETING OF THE TEXAS SECTION

The annual meeting of the Texas Section of the Mathematical Association of America was held at San Marcos, Texas, on April 20–21, 1956, sponsored by Southwest Texas State Teachers College. Sessions were held Friday afternoon and Saturday morning, Professor M. E. Mullings presiding. At the banquet Friday night Dr. Don Cowan of Texas Christian University gave a speech entitled "Education for the Nuclear Age."

There were 108 persons in attendance and 78 of these were members.

At the business meeting the following officers were elected for the coming year: Chairman, Professor Don Cude, Southwest Texas State Teachers College; Vice-Chairman, Professor Martin Wright, University of Houston; Secretary-Treasurer, Professor C. R. Sherer, Texas Christian University.

The program consisted of the following papers:

1. *Algebraic properties of classes of functions*, by Mr. B. T. Goldbeck and Professor A. A. Grau, The University of Oklahoma, presented by Mr. Goldbeck.

Any function of a complex variable, $f(x)$, has a unique expansion in terms of functions of type (n, k) for each n, $k=0, 1, 2, \cdots, n-1$, where a function of type (n, k) is a function $f(x)$ having the property that $f(\omega x)=\omega^k f(x)$, ω being a primitive nth root of unity. Furthermore the classes of functions, (n, k), $k=0, 1, 2, \cdots, n-1$, form a ring isomorphic to the ring of the residue classes of integers, modulo n.

2. *The singularities of a function defined by a Taylor series with gaps*, by Mr. Alan Wilson, The Rice Institute, introduced by the Secretary.

J. M. Whittaker proved that if $f(z)=\sum_{n=1}^{\infty} a_{\lambda_n} z^{\lambda_n} (a_{\lambda_n} \neq 0)$ with radius of convergence one has exactly k isolated singularities (poles or essential) on $|z|=1$ and no other singularities there, then $\overline{\lim}_{n\to\infty} \lambda_n/n \leq k$. S. Mandelbrojt presented without proof essentially the following improvement of Whittaker's result: let a be the length of the smallest arc of $|z|=1$ which contains the k singularities of $f(z)$ and put $\omega=[2\pi/a]$. Then for each integer $p(0 \leq p<\omega)$ the sequence $\{\mu_n^{(p)}\}$ consisting

of those $\lambda_n \equiv p \pmod{\omega}$ satisfies $(1/\omega)\ \overline{\lim}_{n\to\infty} \mu_n^{(\omega)}/n \leqq k$. The present paper contains a proof of the latter theorem along lines suggested by Mandelbrojt.

3. *Euler's diagrams showing all 2^n-minterms*, by Professor R. E. Greenwood, The University of Texas.

The familiar Euler diagram has three intersecting circles showing all $2^3=8$ distinct intersections or regions into which objects can be classified as to whether or not they possess three different attributes. These distinct intersections or regions have been named as minterms by Serrell. It is possible to modify Euler's diagram for case of four (or a higher number of) attributes, but the symmetry disappears. However, Euler's circles may be replaced by a quadrant diagram in which symmetry is retained, particularly for an even number of attributes. Some properties of the quadrant diagram were discussed.

4. *Extremal elements of a convex cone of concave functions*, by Professor E. K. McLachlan, Baylor University.

Let D_n be a convex compact subset of Euclidean n-space, E^n, that contains a sphere of E^n. Let C_n be the set of all real valued nonnegative concave functions f on D_n. A function f is *concave* on D_n if, for x and y in D_n, $f(tx+(1-t)y) \geqq tf(x)+(1-t)f(y)$ for $0 \leqq t \leqq 1$. The set C_n is closed under addition and positive scalar multiplication and hence C_n is a convex cone. A function $f \neq 0$ is an *extremal element* of C_n if $f=f_1+f_2$ where f_1 and f_2 belong to C_n implies that f_1 (and f_2) is proportional to f. The extremal elements of C_1 were characterized. Partial results were given for $n \geqq 2$.

5. *Special series expansions of three Jacobian inverse elliptic functions*, by Mr. R. P. Kelisky, University of Texas.

Power series expansions in x of $\mathrm{sn}^{-1}(x, k)$, and $\mathrm{tn}^{-1}(x, k)$ are obtained in which the coefficients are expressed in terms of the Legendre polynomial $P_n(\lambda)$ where λ is a function of k and the complementary modulus k'. These infinite series are generalizations of known series for $\sin^{-1} x$, $\cos^{-1} x$, and Gregory's series for $\tan^{-1} x$. For $x=1$, series expansions of the complete elliptic integral K in terms of Legendre polynomials may be obtained. In obtaining these expansions, several known formulas relating the complete elliptic integrals E and K to the Legendre functions of orders $1/2$ and $-1/2$ are easily obtained.

6. *Concerning two theorems of R. L. Moore*, by Mr. J. R. Boyd, Chance Vought Aircraft, Dallas.

The primary purpose of this paper is to prove the theorem: *if a collection of fundamental sequences of regions has the hypothesis property, then a T-space satisfies R. L. Moore's Axioms* 0, 1(1), 1(2) *and* 1(3).

With point and region as undefined terms, a *T-space* is a collection of points and regions satisfying Axiom 0 and Theorem 2 as given in R. L. Moore's *Foundations of Point Set Theory*, New York, A.M.S., 1932. A *collection of fundamental sequences of regions* is the collection of regions satisfying Theorem 2 for each point of the space. The definition of *hypothesis property* is a modification of the hypothesis of Theorem 17′ of the same publication.

7. *Spirals and their whirl points and windings*, by Professor R. L. Moore, The University of Texas. (By invitation).

Definitions of the whirl points and windings of spirals in the plane were given.

8. *Growth of entire functions*, by Professor G. R. MacLane, The Rice Institute.

9. *A non-modular compact connected topological lattice*, by Professor D. E. Edmondson, Southern Methodist University.

An example is constructed of a compact connected topological lattice which is not modular. This is done by suitably defining an order on a portion of the unit cube.

10. *Geometric interpretations related to the mean value theorem*, by Dr. R. C. Osborn, The University of Texas.

In differential calculus, insufficient attention is given to geometric interpretations of the Mean Value Theorem (hereafter called MVT). The so-called generalized MVT may be obtained as a specialized form of the ordinary MVT by taking the function in parametric form—a fact which is not new, but which is insufficiently emphasized. The MVT may be taught more meaningfully if the generating function for the MVT is interpreted geometrically. Many generating functions are available, each with its own interpretation, and they lead to various forms of the MVT, some of which are unusual.

11. *Operational analysis of certain nonlinear systems*, by Professor W. T. Guy, Jr., The University of Texas.

Certain solutions of the differential equations of van der Pol and of Duffing are obtained using integral transform techniques under rather mild hypotheses regarding the form of the solution.

12. *On the contingent of G. Bouligand at the points of a line*, by Professor H. C. Parrish, North Texas State College.

If E is a planar set, the *contingent* $T(A, E)$ of E at a point A consists of the set of all rays (A, θ), of origin A in direction θ, such that every open sector $S(A; \theta)$, of vertex A and containing θ, contains a point of E. It is shown that if E is an arbitrary set in a half-plane K bounded by a line L, then at every limit point A of E in L not belonging to a set which is exhaustible and of measure 0, either $T(A, E)$ is the line L (both directions) or $T(A, E)$ is the half-plane K (all directions in K). Several partial converses of this result are given.

13. *Approximation of functions which have harmonic support*, by Mr. Guy Johnson, Jr., The Rice Institute.

Consider a real valued function $v(z)$ defined in a domain D of the complex z-plane. Define the class (hs), functions which have harmonic support in D, to be functions v such that for each $z_0 \in D$, $\exists h(z)$ harmonic and single valued in D satisfying $h(z_0) = V(z_0)$ and $h(z) \leqq v(z)$ in D. The following theorem is proved. If $v \in (hs)$ in D then $\exists$ a sequence of functions $\{u_n\}$, where each u_n is in (hs) and infinitely differentiable, such that $u_n \to v$ subuniformly in D.

14. *College mathematical training for students specializing in agriculture*, by Professor W. I. Layton, Stephen F. Austin State College.

The study included seventy-five colleges that offer training in agriculture. These colleges were of varying size and type in forty states. Requirements for agricultural engineers were excluded, but requirements were included for those preparing for Smith-Hughes work in agriculture and all others who major or minor in agriculture.

Recommendations of departments of agriculture indicated that there is a growing interest in requiring agricultural mathematics instead of college algebra and/or trigonometry in the curricula for college students specializing in agriculture.

15. *Texas Tech's special committee for the advancement of the study of the sciences*, by Professor J. W. Lindsay, Texas Technological College.

At Texas Technological College, the Special Committee for the Advancement of the Study of the Sciences was appointed to study the current scientific manpower shortage in the U. S. The Committee's recommendations will help determine the College's role in stimulating more interest in the study of mathematics and the natural sciences and improving the quality of science teaching in these subjects. The shortage of teachers and mathematicians was illustrated.

16. *Revisions in the 1956 engineering curricula at Texas Tech*, by Professor E. R. Heineman, Texas Technological College.

Entering engineering students will take a placement test in mathematics. Those making unsatisfactory scores will take a non-credit course in intermediate algebra. Those making satisfactory scores (roughly, the highest 40%) will take a dual course (algebra-trigonometry). For the first half of the fall semester, they will take a three-credit course in algebra meeting six times a week. In the second half, trigonometry will be given in the same way. The spring semester's work consists of a three-credit course in analytic geometry, followed by a three-credit calculus course. Two more three-credit courses in calculus and one in differential equations follow.

17. *Types of mathematics used in aeronautical engineering*, by Dr. Nat Edmonson, Jr., Convair, Fort Worth.

The kinds of mathematical analysis in daily use in the research and engineering design groups of a large airplane company engaged in the development and production of weapons systems were described by means of slides and examples. It was pointed out that such fields of mathematics as ordinary and partial differential equations, matrices and determinants, tensor analysis, integral equations, statistical analysis, and numerical analysis are necessary tools in the development of the highly complex modern airplane.

C. R. SHERER, *Secretary*

THE APRIL MEETING OF THE UPPER NEW YORK STATE SECTION

The twelfth annual meeting of the Upper New York State Section of the Mathematical Association of America was held at Alfred University, Alfred, New York, on April 28, 1956. The Chairman of the Section, Professor C. E. Rhodes of Alfred University, presided at the morning session, and the Vice-Chairman, Professor A. J. Coleman of the University of Toronto, presided at the afternoon session.

There were 97 persons in attendance, including 63 members of the Association.

At the business meeting the following officers were elected: Chairman, Professor A. J. Coleman, University of Toronto; Vice-Chairman, Col. W. W. Bessell, Jr., U. S. Military Academy; Secretary, Professor N. G. Gunderson, University of Rochester.

The morning session was devoted to a panel discussion on "The Secondary School—College Transition." Professor A. J. Coleman, University of Toronto, was moderator, and the members of the panel were: Professor H. M. Gehman, University of Buffalo; Mr. C. W. Pflaum, John Marshall High School, Roches-

ter; Dean R. S. Fisk, School of Education, University of Buffalo; Professor C. W. Munshower, Colgate University.

The following papers were presented at the afternoon session:

1. *Mathematics in industry*, by Dr. T. C. Fry, Bell Telephone Laboratories. (By Invitation.)

The substance of this talk appeared under the title, *Mathematics as a profession today in industry*, in this MONTHLY, vol. 63, 1956, pp. 71–80.

2. *A fundamental limit and its applications*, by Professor M. J. Pascual, Siena College.

An elementary geometric proof that the limit of the ratio of arc length to chord length, as each approach zero, is one for smooth curves, was given. This limit was then used to establish several formulas of elementary calculus, which are usually derived by more involved means.

3. *Some special determinants*, by Professor F. R. Olson, University of Buffalo.

Let $x^{(n)}=x(x-1)(x-2)\cdots(x-n+1)$. The Stirling numbers of the first and second kind, S_j^n and T_j^n, are defined as the coefficients in the expansions

$$x^{(n)}=\sum_{j=1}^{n} S_j^n x^j \quad \text{and} \quad x^n=\sum_{j=1}^{n} T_j^n x^{(j)}.$$

It was shown that

$$\sum_{j=1}^{n} S_i^j T_k^{n-1+j}=\begin{cases}0, & k<i\\ i^{n-1}, & k=i,\end{cases}$$

and as a consequence that the determinant

$$\left|T_k^{n-1+j}\right| = (n!)^{n-1}, \qquad j,k=1,2,\cdots,n.$$

This parallels the known result

$$\left|S_k^{n-1+j}\right| = [(n-1)!]^n.$$

4. *False solutions in elementary extremum problems*, by Professor C. S. Ogilvy, Hamilton College.

Examples from elementary calculus were used to illustrate "false" solutions of two types. (1) A maximum is sought and only a minimum is delivered by the machinery of differentiation, because of an unsuitable choice of independent variable. (2) Two "solutions" appear, but one is extraneous, even though both may lie within the admissible range of the problem. The false solution here usually has an interesting geometric interpretation. In general, if two problems requiring different answers lead at some stage to the same equation, then this equation must deliver at least two solutions, one of which will be extraneous to each problem.

5. *Mathematics for statisticians*, by Professor J. E. Freund, Virginia Polytechnic Institute. (By invitation.)

The speaker, who is Chairman of the Section on Training of the American Statistical Association, discussed some of the problems of planning graduate curricula in statistics with special reference to the type of mathematics courses to be included in such programs. He also illustrated briefly what topics in modern algebra and advanced calculus are essential for an M.S. degree in statistics.

N. G. GUNDERSON, *Secretary*

THE MAY MEETING OF THE INDIANA SECTION

The thirty-third annual meeting of the Indiana Section of the Mathematical Association of America was held jointly with the Indiana Council of Teachers of Mathematics at Wabash College, Crawfordsville, Indiana, on May 5, 1956. Professor R. O. Virts of Central High School and the Purdue University Center at Fort Wayne, Chairman of the Section, presided at the general session and at the sectional meetings of the Association which followed.

Professor K. O. May, Chairman of the Mathematics Department of Carleton College, guest speaker for the hour lecture, addressed the general session on the topic, "A Modern Introduction to Mathematics."

There were 133 in attendance including 51 members of the Association.

The following officers were elected: Chairman, Professor C. H. Brumfiel, Ball State Teachers College; Vice-Chairman, Professor C. B. Gass, DePauw University; Secretary-Treasurer, Professor J. C. Polley, Wabash College.

Professor P. D. Edwards, Chairman of the Committee on Awards, reported that five Association medals had been awarded during the year for high mathematical achievement in the Indiana Science Talent Search.

The following short papers were presented:

1. *Non-linear recurrence relations for classical orthogonal polynomials*, by Professor M. S. Webster, Purdue University.

Some of the known and some new non-linear relations involving Hermite, generalized Laguerre, and ultraspherical polynomials were given. A typical proof was presented showing that such a relation essentially characterizes polynomials of the type given.

2. *A geometry experiment*, by Professor C. F. Brumfiel, Ball State Teachers College, and Professor M. E. Shanks, Purdue University, presented by Professor Shanks.

The purpose of this paper was to call the attention of college teachers to shortcomings in high school geometry texts and to encourage the writing of better texts. Defects in Euclid's geometry have long been known but current texts do not remove these flaws. In fact, they add errors of their own. Besides being incomplete, these texts are lacking in logic and give as postulates what are really theorems or definitions. A critique of Euclid's postulates was given and Hilbert's postulates were described briefly. A modified version of Hilbert's postulates was described, a version which is being taught at the Burris High School in Muncie, Indiana in a course for which the textual material has been prepared by the authors of the paper.

3. *Discontinuities for the classroom*, by Professor H. L. Hunzeker, DePauw University.

Graphical illustrations of discontinuous functions for classroom use were obtained from considering common machines as mathematical systems.

4. *An operator identity for* $(D^2+a^2)y=f(x)$, by Professor R. E. MacKenzie, Indiana University, introduced by Professor H. E. Wolfe.

A solution of the differential equation $(D^2+a^2)y=f(x)$ by quadratures was effected by means of an operator identity using trigonometric functions.

5. *On a function defined by means of an infinite radical*, by Professor G. N. Wollan and Mr. D. M. Mesner, Purdue University Center, Fort Wayne, presented by Professor Wollan.

If $0<x\leqq 1$, then x has a non-terminating binary representation $x=a_1a_2\cdots a_n\cdots$. Let $\alpha_n=(-1)^{a_n}$ and $f_n(x)=\sqrt{k+\alpha_1\sqrt{k+\alpha_2\sqrt{k+\cdots+\alpha_n\sqrt{k}}}}$, $n=1, 2, 3, \cdots$, and let I denote the interval $0<x\leqq 1$. When $k>2+\sqrt{2}$, then $\lim_{n\to\infty} f_n(x)$ exists and is real for every x in I and this defines a function $f(x)=\lim_{n\to\infty} f_n(x)$. This function is discontinuous at every point in I having a terminating binary representation and is continuous elsewhere in I. The function is not monotone in any sub-interval of I and yet has a derivative equal to zero at each point of a dense set of points of I and has a left derivative equal to zero at every point of discontinuity.

J. C. POLLEY, *Secretary*

THE MAY MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The spring meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the U. S. Naval Academy, Annapolis, Maryland, on May 5, 1956. Professor F. E. Johnston, Chairman of the Section, presided at the morning and afternoon sessions.

There were 99 persons in attendance, including 75 members of the Association.

The following officers were elected to serve for a period of one year: Chairman Professor R. C. Yates, College of William and Mary; Vice-Chairmen, Professor J. E. Freund, Virginia Polytechnic Institute and Professor D. B. Lloyd, District of Columbia Teachers College; Secretary, Professor R. P. Bailey, U. S. Naval Academy; Treasurer, Professor T. W. Moore, U. S. Naval Academy.

The following papers were presented.

1. *The coloring of maps*, by Professor R. W. Rector, U. S. Naval Academy.

The speaker presented a historical survey of the four color problem with particular attention to the method of Birkhoff and Lewis for constrained chromatic polynomials for the n-ring. Certain steps were indicated leading to the solution of the constrained chromatic polynomials of the seven-ring.

2. *Aliquot sequences (preliminary report)*, by Mr. G. A. Paxson, U. S. Army, Fort Meade, Maryland.

Let $S(n)=\sum_{d|n} d$, $0<d<n$. $S(n)$ is called the sum of the *aliquot parts* of n. Put $S^0(n)=n$, $S^1(n)=S(n)$, and $S^{k+1}(n)=S(S^k(n))$. The infinite sequence n, $S(n)$, $S^2(n)$, $\cdots$ is called the *aliquot sequence* with leader n.

A sequence is *eventually periodic* if any term of the sequence recurs.

Conjecture: Every sequence is eventually periodic. Every sequence examined for this property, except a few, displayed it upon computation of a sufficient number of terms. The few involve terms of great size, rendering continued hand computation impracticable. All sequences with leaders $n\leqq 10{,}000$ are being computed until discovered to be periodic or until a term with more than 20 digits is reached. Computation is being done on the IBM Type 650 Magnetic Drum Data-Processing Machine at the Watson Scientific Computing Laboratory, 612 West 116th Street, New York, New York.

3. *A graphical method of determining the instantaneous center of zero acceleration of a rigid body having plane motion*, by Professor J. P. Hoyt, U. S. Naval Academy.

If the acceleration of one point on a rigid body having plane motion is known together with the angular velocity and angular acceleration of lines in the body, the instantaneous center of zero acceleration is found as the intersection of two circles. A property of these circles is considered for a special case. Certain general results and applications are discussed.

4. *A set of predictor and corrector formulas for the numerical solution of ordinary differential equations*, by Dr. C. H. Frick and Mr. J. E. Mulligan, Jr., U. S. Naval Proving Ground, Dahlgren, Virginia, presented by Mr. Mulligan.

The use of

$$(1) \qquad hy'_n = 12y_{n-2} - 12y_{n-1} + h(5y'_{n-2} + 8y'_{n-1}) + \tfrac{1}{3}h^4y^{iv}$$

or

$$(2) \qquad hy'_n = 33y_{n-2} + 24y_{n-2} - 57y_{n-1} + h(10y'_{n-2} + 57y'_{n-2} + 24y'_{n-1}) + \frac{33}{180}h^6y^{vi}$$

for predicting the slope, and

$$(3) \qquad y_n = 0.2y_{n-2} + 0.8y_{n-1} + h(0.8y'_{n-1} + 0.4y'_n) - \frac{1}{30}h^4y^{iv}$$

or

$$(4) \qquad y_n = 0.1y_{n-2} + 1.8y_{n-2} - 0.9y_{n-1} + h(0.9y'_{n-2} + 1.8y'_{n-1} + 0.3y'_n) - \frac{1}{200}h^6y^{vi}$$

for estimating y_n was discussed. The estimated value of y_n can be entered into the differential equation $y' = f(x, y)$ and corrected values of y' and y computed.

5. *A Carnegie Foundation project in modern mathematics for college sophomores (preliminary report)*, by Professor R. V. Andree, University of Oklahoma and Haverford College.

The Carnegie Corporation is sponsoring an experiment in the inclusion of some mathematics other than calculus in the college sophomore course and the inclusion of examples from fields other than physics. It is suggested that about three-fourths of the academic year be spent on calculus, utilizing the (polynomial) calculus contained in many freshman courses. The remaining one-fourth academic year will be spent on material mainly from abstract algebra.

6. *The national status of mathematics contests in secondary schools*, by Professor D. B. Lloyd, District of Columbia Teachers College.

The purposes of this study were two-fold: (1) to survey the mathematics contests now being given in high schools of the United States; and (2) to ascertain the prevailing sentiment for or against the launching of a nation-wide contest. Some 125 questionnaires were distributed to all states through state representatives and others prominent in mathematics education. Thirty-six contests were discovered in 26 states, involving some 3900 schools. As to the desirability of a single nation-wide contest, the enormity of the problem, and the impracticability of attempting to adapt a single contest to the heterogeneous local needs, were cited from some areas. However, some 22 states reported no contests at all,—thus indicating fertile virgin soil for introducing a well-planned contest program under the auspices of a national body. The study revealed the feasibility of a nationally sponsored contest, provided that interference with certain local contests was carefully avoided.

7. *Commutativity of finite matrices*, by Dr. Olga Taussky-Todd, National Bureau of Standards, introduced by the Secretary.

Various generalizations of the concept of commutativity of a pair A, B of finite matrices are being discussed. One such generalization consists in assuming that all polynomials $p(A, B)$ have as eigenvalues $p(\alpha_i, \beta_i)$ where α_i, β_i are the eigenvalues of A and B taken in a special order. A more recent generalization assumes that only all linear combinations $\lambda A+\mu B$ have as eigenvalues $\lambda\alpha_i+\mu\beta_i$. A further generalization is obtained by studying the matrices in the pencil $\lambda A+\mu B$ which have multiple characteristic roots. The vanishing of the higher commutators of A and B also plays a big role. If in particular $B=A^*$ (the complex conjugate and transpose of A) then the vanishing of the higher commutators sheds light on the nature of A itself.

R. P. BAILEY, *Secretary*

THE MAY MEETING OF THE ROCKY MOUNTAIN SECTION

The thirty-ninth annual meeting of the Rocky Mountain Section of the Mathematical Association of America was held at the University of Utah, Salt Lake City, Utah, on May 4 and 5, 1956. Professor C. R. Wylie, Jr., Chairman of the Section, presided at all three sessions.

There were 77 persons registered for the meeting, including 48 members of the Association.

Officers elected at the meeting for 1956–1957 were: Chairman, Professor R. R. Gutzman, Colorado School of Mines; Vice-Chairman, Professor J. S. Leech, Colorado College; Secretary-Treasurer, Professor F. M. Carpenter, Colorado School of Mines.

The following papers were presented:

1. *Connectedness in partially ordered sets*, by Professor L. E. Ward, Jr., University of Utah.

A partially ordered space is a triple $(X, T, <)$ where (X, T) is a topological space, $(X, <)$ is a partially ordered set, and there obtains some harmonious relation between T and $<$. Toward the end of characterizing various partially ordered spaces theoretically, results of the following type are of interest: order hypothesis→topological conclusion. Sample theorem: if the set of $a \leqq x$ is compact for each $x \in X$, if the partial order is continuous and dense, and if each pair of points of X have a common predecessor, then X is connected.

2. *Convergence of sequences of linear fractional transformations*, by Professor W. J. Thron, University of Colorado.

Let $\{T_n(z)\}$ be a sequence of nonsingular linear fractional transformations. Let Z be the set in the complex plane, such that $\{T_n(z)\}$ converges for all $z \in Z$. Finally, let $T(z)$ be the function to which the sequence converges on Z. Possible domains of convergence Z and corresponding limit function $T(z)$ are determined. Among other results it is proved that $T(z)$ must either be a constant for all $z \in Z$, or a constant for all but one value of Z, or a linear fractional transformation (in this case Z is the whole plane).

3. *Orthonormal functions used for the approximate solution of integro-differential equations*, by Professor F. M. Stein, Colorado Agricultural and Mechanical College.

Under certain conditions the integro-differential equation $U(u)=L(u)-\int_a^b h(x, t)u(t)dt=f(x)$

with boundary conditions $U_i(u)=0 (i=1, 2, \cdots, m)$ has a unique solution of the form $u(x)=\int_a^b H(x, t)f(t)dt$, where $H(x, t)$ is the Green's function of the problem. Let $S_n(x)$ be a sum of functions of the complete set $\phi_i(x)$ which are orthonormal with respect to the weight function $\rho(x)$ and which satisfy the boundary conditions. Under the criterion that $\int_a^b |f(x)-U[S_n(x)]|^r dx$ shall be least for $r>0$, this paper examines the sufficient condition that $S_n(x)$ as well as its first m derivatives be uniformly convergent to the corresponding derivative of the solution $u(x)$ as n becomes infinite.

4. *The Dirichlet series transformation*, by Professor J. R. Britton, University of Colorado.

Let $F(t)$ be a complex valued function defined for nonnegative integral values of t. If a is a positive constant, the Dirichlet series $\sum_{t=0}^{\infty} a^{-st}F(t)$ is the Dirichlet series transform, $D\{F(t)\}$. The series converges, for example, if $F(t)=\theta(k^t)$, $k>0$, $t>t_0$. Simple properties of the transform were developed, in particular, the convolution theorem

$$[D\{F(t)\}][D\{G(t)\}]=D\left\{\sum_{\tau=0}^{t} F(t-\tau)G(\tau)\right\}.$$

Application was made to "summation" equations of the convulution type $\sum_{\tau=0}^{t} F(t-\tau)G(\tau)=H(t)$, where $G(t)$ and $H(t)$ are given functions, and $F(t)$ is to be found.

5. *Series solution of linear differential equations*, by Mr. C. A. Grimm, South Dakota School of Mines and Technology.

By the use of the exponential shift formula,

$$P(\Delta)e^{mt}f(t)=e^{mt}P(\Delta+m)f(t)$$

and the substitution $x-a=e^t$, one easily arrives at the formula $P(\Delta)(x-a)^m=P(m)(x-a)^m$. Then by writing (if possible) a linear differential equation in the form $f(\Delta)y+(x-a)^k g(\Delta)y=0$ with the assumed solution $y=\sum_{n=0}^{\infty} c_n(x-a)^{\lambda+n}$, $c_0\neq 0$, the recurrence relation, standardly found by the method of Frobenius, is readily obtained.

6. *Homogeneous continua*, by Professor C. E. Burgess, University of Utah.

A brief history of results on homogeneous continua was given, and some related unsolved problems were mentioned.

7. *Finite geometries and difference sets*, by Professor B. W. Jones, University of Colorado.

Cyclic finite geometries were defined and it was shown that any such geometry leads to a perfect difference set. Conversely, any perfect difference set leads to a finite geometry. One or two generalizations were indicated.

8. *Decimal expansions*, by Dr. Bodo Volkmann, University of Mainz, Visiting Professor, University of Utah. (By invitation.)

Borel proved in 1909 that almost all real numbers are normal, *i.e.*, their decimal expansion involves each of the possible digits with the same asymptotic frequency. A number of generalizations of Borel's theorem to the digit distribution of non-normal numbers were obtained by Besicovitch, Knichal, Eggleston, and the speaker. These theorems describe certain sets of real numbers in terms of their Hausdorff (or fractional) dimension. Among the unsolved problems on decimal expansions are questions such as whether numbers like e, π, or $\sqrt{2}$ are normal in any scale, and what conclusions can be drawn from the normality of a real number in a given scale about its digits distribution in any other scale.

9. *Some constants associated with the Riemann zeta-function*, by Dr. W. E. Briggs, University of Colorado.

The Riemann zeta-function has the representation

$$\zeta(s) = \frac{1}{s-1} + \sum_{0}^{\infty} \frac{(-1)^n \gamma_n}{n!} (s-1)^n$$

where

$$\gamma_n = \lim_{N \to \infty} \left[\sum_{l=1}^{N} \frac{\log^n l}{n} - \int_1^N \frac{\log^n x}{x}\, dx \right]$$

and satisfies the functional equation $\zeta(s) = 2^s \pi^{s-1} \sin s\pi \Gamma(1-s)\zeta(1-s)/2$. By letting $s = 1-2m$, where m is a positive integer, it can be shown that infinitely many γ_n are positive, and infinitely many are negative. From a representation of γ_n as an infinite series, it can be shown that $\gamma_n = O\{(n/2e)^n\}$. Using the fact that $\zeta(s) - 1/(s-1)$ is an entire function of order one, it can be shown that if $\epsilon > 0$, then $n^{-\epsilon n} < |\gamma_n| < n^{\epsilon n}$ is true for infinitely many n.

10. *Production delay period*, by Professor E. A. Davis, University of Utah.

In the production of economic goods a "production delay period" elapses between the time factors are joined in a firm's productive mechanism and the time resulting output materializes. The delay period, "time", acts somewhat as a productive factor whose cost is expressed through the interest rate. The author considered adjustments of a firm attempting to maximize "profit" integrals, strict competition being assumed, under the assumption that the production delay period for the firm was: a) of zero length; b) of fixed length; c) of length subject to entrepreneur control. The latter two cases lead naturally to a theory involving producer expectations.

11. *Summing of series with missing terms*, by Dr. H. J. Fletcher, Brigham Young University.

A method is presented to sum many series of the form $\sum a_n b_n$ where the series $\sum a_n$ is known and b_n is a periodic function of n. The method consists of expressing b_n as a finite sum of sines and cosines, and summing the corresponding trigonometric series. In particular, if the series $\sum a_n \sin nx$ can be summed to a known function, then the corresponding sum which has terms missing periodically can be summed to a known function.

12. *Fitting empirical equations to data from small orifice fluid meters*, by Professor S. R. Smith, University of Wyoming.

Two empirical equations, $C = R/(a+bR)$ and $C = d + k/\sqrt{R}$ were fitted to the orifice meter data (C, R data was determined by Professor Eric Lindahl, University of Wyoming). Water was the fluid used in each case. C is the coefficient of discharge of the meter and R its corresponding Reynolds number, both dimensionless.

13. *Errors in linear systems*, by Professor C. A. Hutchinson, University of Colorado. (Presented by title.)

14. *Coordination of college and high school mathematics topics*, by Professor O. M. Rasmussen, University of Denver.

The author expressed the need for coordination of mathematics programs in the high schools and colleges, especially with respect to topics of modern mathematics. This need was shown by a brief review of some of the present activity concerning modern topics in mathematics. It was emphasized that coordination should take place on national, regional, and local levels with each individual aiding in any manner possible.

F. M. CARPENTER, *Secretary*

CALENDAR OF FUTURE MEETINGS

Fortieth Annual Meeting, University of Rochester, Rochester, New York, December 29, 1956.

Thirty-eighth Summer Meeting, Pennsylvania State University, University Park, Pennsylvania, August 26–27, 1957.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Westinghouse Research Laboratories, Pittsburgh, Pennsylvania, Spring, 1957.

ILLINOIS, Illinois State Normal University, Normal, May 10–11, 1957.

INDIANA, May 4, 1957.

IOWA, Iowa State Teachers College, Cedar Falls, April 26–27, 1957.

KANSAS

KENTUCKY, Berea College, Berea, April 20, 1957.

LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 15–16, 1957.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, College of William and Mary, Williamsburg, Virginia, December 1, 1956.

METROPOLITAN NEW YORK

MICHIGAN, Wayne University, Detroit, March 23, 1957.

MINNESOTA, Concordia College, Moorhead, October 6, 1956.

MISSOURI, Southeast Missouri State College, Cape Girardeau, Spring, 1957.

NEBRASKA, University of Nebraska, Lincoln, April 27, 1957.

NORTHEASTERN, University of Connecticut, Storrs, November 24, 1956.

NORTHERN CALIFORNIA, University of California, Berkeley, January 12, 1957.

OHIO

OKLAHOMA, Oklahoma City University, October 26, 1956.

PACIFIC NORTHWEST, Oregon State College, Corvallis, June, 1957.

PHILADELPHIA, Muhlenberg College, Allentown, Pennsylvania, November 24, 1956.

ROCKY MOUNTAIN, Colorado School of Mines, Golden, Spring, 1957.

SOUTHEASTERN, Emory University, Emory University, Georgia, March 15–16, 1957.

SOUTHERN CALIFORNIA, San Diego State College, Spring, 1957.

SOUTHWESTERN, University of Arizona, Tucson, April, 1957.

TEXAS, University of Houston, Houston, April, 1957.

UPPER NEW YORK STATE, Skidmore College, Saratoga Springs, May 4, 1957.

WISCONSIN, Wisconsin State College, Whitewater, May, 1957.

THE CARUS MATHEMATICAL MONOGRAPHS

A new Carus Monograph has been published in 1956. The title of this Monograph is:

MONOGRAPH 11: *Irrational Numbers,* by Ivan Niven.

Each member of the Association may purchase one copy of this Carus Monograph at the special price of $1.75. Orders should be addressed to: Mathematical Association of America, University of Buffalo, Buffalo 14, New York.

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